SECTION 2.6: THE SQUEEZE (SANDWICH) THEOREM

LEARNING OBJECTIVES

• Understand and be able to rigorously apply the Squeeze (Sandwich) Theorem when evaluating limits at a point and "long-run" limits at $(\pm)$ infinity.

PART A: APPLYING THE SQUEEZE (SANDWICH) THEOREM TO LIMITS AT A POINT

We will **formally state** the Squeeze (Sandwich) Theorem in Part B.

Example 1 below is one of many basic examples where we use the Squeeze (Sandwich) Theorem to show that $\lim_{x \to 0} f(x) = 0$, where $f(x)$ is the **product** of a **sine or cosine expression** and a **monomial of even degree**.

• The idea is that "something approaching 0" times "something bounded" (that is, trapped between two real numbers) will approach 0. Informally,

$$\left(\text{Limit Form} \quad 0 \cdot \text{bounded}\right) \Rightarrow 0 .$$

Example 1 (Applying the Squeeze (Sandwich) Theorem to a Limit at a Point)

Let $f(x) = x^2 \cos\left(\frac{1}{x}\right)$. Prove that $\lim_{x \to 0} f(x) = 0$.

§ Solution

• We first **bound** $\cos\left(\frac{1}{x}\right)$, which is **real** for all $x \neq 0$.

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \quad (\forall x \neq 0) \quad \Rightarrow$$

• **Multiply** all three parts by x^2 so that the middle part becomes $f(x)$.

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2 \quad (\forall x \neq 0) \quad \Rightarrow$$

WARNING 1: We **must** observe that $x^2 > 0$ for all $x \neq 0$, or at least on a **punctured neighborhood** of $x = 0$, so that we can multiply by x^2 **without reversing** inequality symbols.

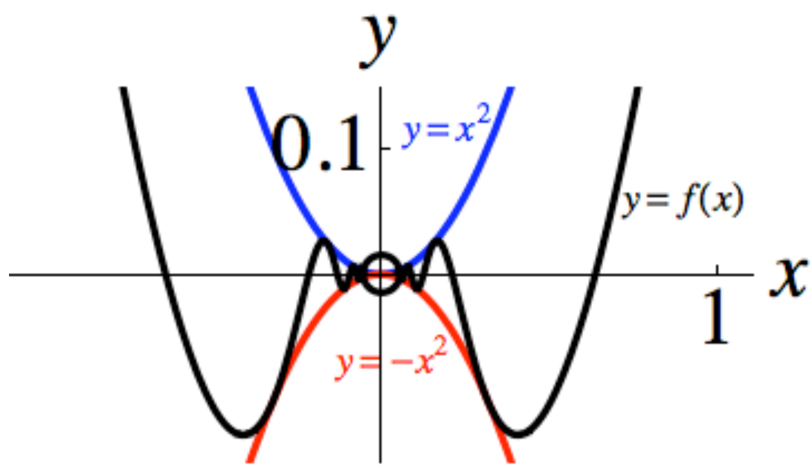
• As $x \to 0$, the **left and right parts approach 0**. Therefore, by the Squeeze (Sandwich) Theorem, the **middle part**, $f(x)$, is **forced to approach 0**, also. The middle part is "squeezed" or "sandwiched" between the left and right parts, so it **must approach the same limit** as the other two do.

$\lim_{x\to 0}\left(-x^2\right) = 0$, and $\lim_{x\to 0} x^2 = 0$, so $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem.

Shorthand: As $x \to 0$,

$$\underbrace{-x^2}_{\to 0} \le \underbrace{x^2 \cos\left(\frac{1}{x}\right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{x^2}_{\to 0} \quad \left(\forall x \ne 0\right).$$

The graph of $y = x^2 \cos\left(\frac{1}{x}\right)$, together with the squeezing graphs of $y = -x^2$ and $y = x^2$, is below.



(The axes are scaled differently.)

§

In Example 2 below, $f(x)$ is the **product** of a **sine or cosine expression** and a **monomial of odd degree**.

Example 2 (Handling Complications with Signs)

Let $f(x) = x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$. Use the Squeeze Theorem to find $\lim_{x \to 0} f(x)$.

§ Solution 1 (Using Absolute Value)

• We first **bound** $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$, which is **real** for all $x \neq 0$.

$$-1 \leq \sin\left(\frac{1}{\sqrt[3]{x}}\right) \leq 1 \quad (\forall x \neq 0) \quad \Rightarrow$$

• **WARNING 2:** The problem with multiplying all three parts by x^3 is that $x^3 < 0$ when $x < 0$. The $\leq$ inequality symbols would have to be **reversed** for $x < 0$.

Instead, we use **absolute value** here. We could write $0 \leq \left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \leq 1 \quad (\forall x \neq 0)$, but we assume that absolute values are **nonnegative**.

$$\left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \leq 1 \quad (\forall x \neq 0) \quad \Rightarrow$$

• **Multiply** both sides of the inequality by $|x^3|$. We know $|x^3| > 0 \quad (\forall x \neq 0)$.

$$|x^3| \left|\sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \leq |x^3| \quad (\forall x \neq 0) \quad \Rightarrow$$

• "The **product** of absolute values equals the **absolute value** of the product."

$$\left|x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)\right| \leq |x^3| \quad (\forall x \neq 0) \quad \Rightarrow$$

• If, say, $|a| \leq 4$, then $-4 \leq a \leq 4$. Similarly:

$$-|x^3| \leq x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \leq |x^3| \quad (\forall x \neq 0) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x \to 0}\left(-\left|x^3\right|\right)=0$, and $\lim_{x \to 0}\left|x^3\right|=0$, so $\lim_{x \to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)=0$ by the Squeeze Theorem.

Shorthand: As $x \to 0$,

$$\underbrace{-\left|x^3\right|}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{\left|x^3\right|}_{\to 0} \quad (\forall x \ne 0).$$ §

§ Solution 2 (Split Into Cases: Analyze Right-Hand and Left-Hand Limits Separately)

First, we analyze: $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$.

Assume $x > 0$, since we are taking a limit as $x \to 0^+$.

• We first **bound** $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$, which is **real** for all $x \ne 0$.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le 1 \quad (\forall x > 0) \quad \Rightarrow$$

• **Multiply** all three parts by x^3 so that the middle part becomes $f(x)$. We know $x^3 > 0$ for all $x > 0$.

$$-x^3 \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le x^3 \quad (\forall x > 0) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x \to 0^+}\left(-x^3\right)=0$, and $\lim_{x \to 0^+} x^3=0$, so $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)=0$ by the Squeeze Theorem.

Shorthand: As $x \to 0^+$,

$$\underbrace{-x^3}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{x^3}_{\to 0} \quad (\forall x > 0).$$

Second, we analyze: $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)$.

Assume $x < 0$, since we are taking a limit as $x \to 0^-$.

• We first **bound** $\sin\left(\frac{1}{\sqrt[3]{x}}\right)$, which is **real** for all $x \neq 0$.

$$-1 \le \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le 1 \quad (\forall x < 0) \quad \Rightarrow$$

• **Multiply** all three parts by x^3 so that the middle part becomes $f(x)$. We know $x^3 < 0$ for all $x < 0$, so we **reverse** the $\le$ inequality symbols.

$$-x^3 \ge x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \ge x^3 \quad (\forall x < 0) \quad \Rightarrow$$

• **Reversing the compound inequality** will make it easier to read.

$$x^3 \le x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \le -x^3 \quad (\forall x < 0) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x \to 0^-} x^3 = 0$, and $\lim_{x \to 0^-} \left(-x^3\right) = 0$, so $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$ by the Squeeze Theorem.

Shorthand: As $x \to 0^-$,

$$\underbrace{x^3}_{\to 0} \le \underbrace{x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right)}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{-x^3}_{\to 0} \quad (\forall x < 0).$$

Now, $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$, and $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$, so $\lim_{x \to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$. §

Example 3 (Limits are Local)

Use $\lim_{x\to 0} x^2 = 0$ and $\lim_{x\to 0} x^6 = 0$ to show that $\lim_{x\to 0} x^4 = 0$.

§ Solution

Let $I = (-1,1)\setminus\{0\}$. I is a **punctured neighborhood** of 0.
Shorthand: As $x \to 0$,

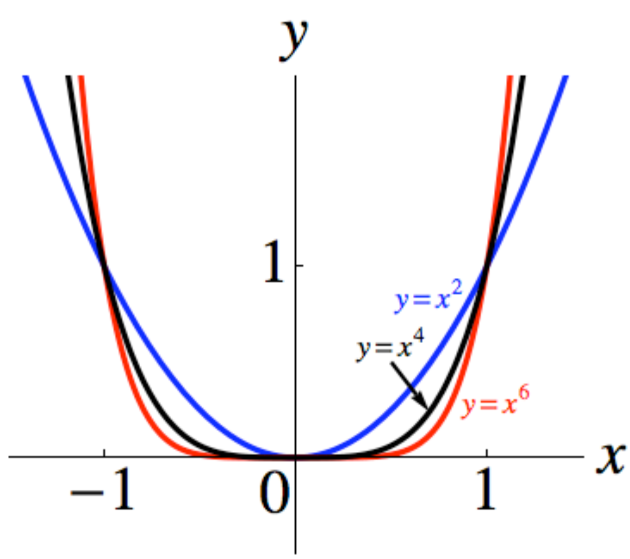
$$\underbrace{x^6}_{\to 0} \le \underbrace{x^4}_{\substack{\text{Therefore, } \to 0 \\ \text{by the Squeeze} \\ \text{(Sandwich) Theorem}}} \le \underbrace{x^2}_{\to 0} \quad (\forall x \in I)$$

WARNING 3: The **direction** of the $\le$ inequality symbols may confuse students. Observe that $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, and $\frac{1}{16} < \frac{1}{4}$.

We conclude: $\lim_{x\to 0} x^4 = 0$.

We do **not** need the compound inequality to hold true for **all** nonzero values of *x*. We only need it to hold true on some **punctured neighborhood** of 0 so that we may apply the Squeeze (Sandwich) Theorem to the **two-sided** limit $\lim_{x\to 0} x^4$. This is because **"Limits are Local."**

As seen below, the graphs of $y = x^6$ and $y = x^2$ squeeze (from below and above, respectively) the graph of $y = x^4$ on *I*. In Chapter 6, we will be able to find the **areas** of the bounded regions.



§

PART B: THE SQUEEZE (SANDWICH) THEOREM

We will call B the "bottom" function and T the "top" function.

The Squeeze (Sandwich) Theorem

Let B and T be functions such that $B(x) \le f(x) \le T(x)$ on a **punctured neighborhood** of a.

If $\lim_{x \to a} B(x) = L$ and $\lim_{x \to a} T(x) = L$ $(L \in \mathbb{R})$, then $\lim_{x \to a} f(x) = L$.

Variation for Right-Hand Limits at a Point

Let $B(x) \le f(x) \le T(x)$ on some **right-neighborhood** of a.

If $\lim_{x \to a^+} B(x) = L$ and $\lim_{x \to a^+} T(x) = L$ $(L \in \mathbb{R})$, then $\lim_{x \to a^+} f(x) = L$.

Variation for Left-Hand Limits at a Point

Let $B(x) \le f(x) \le T(x)$ on some **left-neighborhood** of a.

If $\lim_{x \to a^-} B(x) = L$ and $\lim_{x \to a^-} T(x) = L$ $(L \in \mathbb{R})$, then $\lim_{x \to a^-} f(x) = L$.

PART C: VARIATIONS FOR "LONG-RUN" LIMITS

In the upcoming Example 4, $f(x)$ is the **quotient** of a **sine or cosine expression** and a **polynomial**.

- The idea is that "something bounded" divided by "something approaching $(\pm)$ infinity" will approach 0. Informally,

$$\left(\text{Limit Form } \frac{\text{bounded}}{\pm\infty} \right) \Rightarrow 0 .$$

Example 4 (Applying the Squeeze (Sandwich) Theorem to a "Long-Run" Limit; Revisiting Section 2.3, Example 6)

Evaluate: a) $\lim_{x\to\infty} f(x)$ and b) $\lim_{x\to-\infty} f(x)$, where $f(x)=\dfrac{\sin x}{x}$.

§ Solution to a)

Assume $x>0$, since we are taking a limit as $x\to\infty$.

- We first **bound** $\sin x$.

$$-1\le \sin x\le 1 \quad (\forall x>0) \quad \Rightarrow$$

- **Divide** all three parts by x $(x>0)$ so that the middle part becomes $f(x)$.

$$-\frac{1}{x}\le \frac{\sin x}{x}\le \frac{1}{x} \quad (\forall x>0) \quad \Rightarrow$$

- Now, apply the **Squeeze (Sandwich) Theorem**.

$\lim_{x\to\infty}\left(-\dfrac{1}{x}\right)=0$, and $\lim_{x\to\infty}\dfrac{1}{x}=0$, so

$\lim_{x\to\infty}\dfrac{\sin x}{x}=0$ by the Squeeze Theorem.

Shorthand: As $x\to\infty$,

$$\underbrace{-\frac{1}{x}}_{\to 0}\le \underbrace{\frac{\sin x}{x}}_{\substack{\text{Therefore, }\to 0\\ \text{by the Squeeze}\\ \text{(Sandwich) Theorem}}}\le \underbrace{\frac{1}{x}}_{\to 0} \quad (\forall x>0).$$ §

§ Solution to b)

Assume $x<0$, since we are taking a limit as $x\to-\infty$.

- We first **bound** $\sin x$.

$$-1\le \sin x\le 1 \quad (\forall x<0) \quad \Rightarrow$$

- **Divide** all three parts by x so that the middle part becomes $f(x)$. But $x<0$, so we must **reverse** the $\le$ inequality symbols.

$$-\frac{1}{x}\ge \frac{\sin x}{x}\ge \frac{1}{x} \quad (\forall x<0) \quad \Rightarrow$$

- **Reversing the compound inequality** will make it easier to read.

$$\frac{1}{x}\le \frac{\sin x}{x}\le -\frac{1}{x} \quad (\forall x<0) \quad \Rightarrow$$

• Now, apply the **Squeeze (Sandwich) Theorem**.

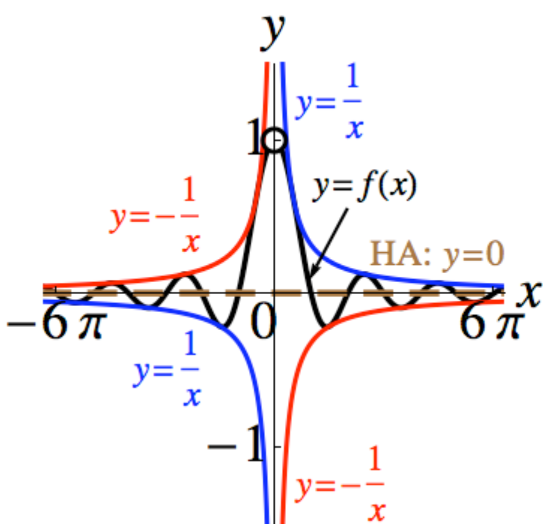
$$\lim_{x\to-\infty}\frac{1}{x}=0, \text{ and } \lim_{x\to-\infty}\left(-\frac{1}{x}\right)=0, \text{ so}$$

$$\lim_{x\to-\infty}\frac{\sin x}{x}=0 \text{ by the Squeeze Theorem.}$$

Shorthand: As $x\to-\infty$,

$$\underbrace{\frac{1}{x}}_{\to 0} \le \underbrace{\frac{\sin x}{x}}_{\substack{\text{Therefore, }\to 0\\ \text{by the Squeeze}\\ \text{(Sandwich) Theorem}}} \le \underbrace{-\frac{1}{x}}_{\to 0} \quad (\forall x<0)$$

The graph of $y=\frac{\sin x}{x}$, together with the squeezing graphs of $y=-\frac{1}{x}$ and $y=\frac{1}{x}$, is below. We can now justify the **HA** at $y=0$ (the x-axis).



(The axes are scaled differently.) §

<u>Variation for "Long-Run" Limits to the Right</u> Let $B(x)\le f(x)\le T(x)$ on some x-interval of the form (c,∞), $c\in\mathbb{R}$. If $\lim_{x\to\infty}B(x)=L$ and $\lim_{x\to\infty}T(x)=L$ $(L\in\mathbb{R})$, then $\lim_{x\to\infty}f(x)=L$.

• In Example 4a, we used $c=0$. We need the compound inequality to hold **"forever" after some point *c***.

<u>Variation for "Long-Run" Limits to the Left</u> Let $B(x)\le f(x)\le T(x)$ on some x-interval of the form $(-\infty,c)$, $c\in\mathbb{R}$. If $\lim_{x\to-\infty}B(x)=L$ and $\lim_{x\to-\infty}T(x)=L$ $(L\in\mathbb{R})$, then $\lim_{x\to-\infty}f(x)=L$.