Prologue: Lebesgue's 1901 paper that changed the integral ... forever

Sur une généralisation de l'intégrale définie

On a generalization of the definite integral[1]
Note by Mr. H. Lebesgue. Presented by M. Picard.

In the case of continuous functions, the notions of the integral and antiderivatives are identical. Riemann defined the integral of certain discontinuous functions, but all derivatives are not integrable in the sense of Riemann. Research into the problem of antiderivatives is thus not solved by integration, and one can desire a definition of the integral including as a particular case that of Riemann and allowing one to solve the problem of antiderivatives.(1) To define the integral of an increasing continuous function

$$y(x) \quad (a \leq x \leq b)$$

we divide the interval (a, b) into subintervals and sums the quantities obtained by multiplying the length of each subinterval by one of the values of y when x is in the subinterval. If x is in the interval (a_i, a_{i+1}), y varies between certain limits m_i, m_{i+1}, and conversely if y is between m_i and m_{i+1}, x is between a_i and a_{i+1}. So that instead of giving the division of the variation of x, that is to say, to give the numbers a_i, we could have given to ourselves the division of the variation of y, that is to say, the numbers m_i. From here there are two manners of generalizing the concept of the integral. We know that the first (to be given the numbers a_i) leads to the definition given by Riemann and the definitions of the integral by upper and lower sums given by Mr. Darboux. Let us see the second. Let the function y range between m and M. Consider the situation

$$m = m_0 < m_1 < m_2 < \cdots < m_{p-1} < M = m_p$$

$y = m$ when x belongs to the set E_0; $m_{i-1} < y \leq m_i$ when x belongs to the set E_i.[2] We will define the measures λ_0, λ_i of these sets. Let us consider one or the other of the two sums

$$m_0\lambda_0 + \sum m_i\lambda_i\,; \quad m_0\lambda_0 + \sum m_{i-1}\lambda_i\,;$$

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[1] This is a translation of Lebesgue's paper where he first reveals his integration theory. This paper appeared in Comptes Rendus de l'Academie des Sciences (1901), pp. 1025–1028, and is translated by Paul Loya and Emanuele Delucchi.

[2] Translator's footnote: That is, Lebesgue defines $E_0 = y^{-1}(m) = \{x \in [a, b]\,;\, y(x) = m\}$ and $E_i = y^{-1}(m_{i-1}, m_i] = \{x \in [a, b]\,;\, m_{i-1} < y(x) \leq m_i\}$.

if, when the maximum difference between two consecutive m_i *tends to zero, these sums tend to the same limit independent of the chosen* m_i*, this limit will be, by definition, the integral of* y*, which will be called integrable.*

Let us consider a set of points of (a, b); one can enclose in an infinite number of ways these points in an enumerably infinite number of intervals; the infimum of the sum of the lengths of the intervals is the measure of the set.[3] A set E is said to be *measurable* if[4] its measure together with that of the set of points not forming E gives the measure of (a, b).(2) Here are two properties of these sets: Given an infinite number of measurable sets E_i, the set of points which belong to at least one of them is measurable; if the E_i are such that no two have a common point, the measure of the set thus obtained is the sum of measures of the E_i. The set of points in common with all the E_i is measurable.[5]

It is natural to consider first of all functions whose sets which appear in the definition of the integral are measurable. One finds that: *if a function bounded in absolute value is such that for any* A *and* B*, the values of* x *for which* $A < y \leq B$ *is measurable, then it is integrable* by the process indicated. Such a function will be called *summable*. The integral of a summable function lies between the lower integral and the upper integral.[6] It follows that *if an integrable function is summable in the sense of Riemann, the integral is the same with the two definitions.* Now, *any integrable function in the sense of Riemann is summable*, because the set of all its points of discontinuity has measure zero, and one can show that if, by omitting the set of values of x of measure zero, what remains is a set at each point of which the function is continuous, then this function is summable. This property makes it immediately possible to form nonintegrable functions in the sense of Riemann that are nevertheless summable. Let $f(x)$ and $\varphi(x)$ be two continuous functions, $\varphi(x)$ not always zero; a function which does not differ from $f(x)$ at the points of a set of measure zero that is everywhere dense and which at these points is equal to $f(x) + \varphi(x)$ is summable without being integrable in the sense of Riemann. *Example:* The function equal to 0 if x is irrational, equal to 1 if x is rational. The above process of construction shows that the set of all summable functions has cardinality greater than the continuum. Here are two properties of functions in this set.

(1) *If* f *and* φ *are summable,* $f + \varphi$ *is and the integral of* $f + \varphi$ *is the sum of the integrals of* f *and of* φ*.*
(2) *If a sequence of summable functions has a limit, it is a summable function.*

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[3] Translator's footnote: Denoting by $\mathfrak{m}^*(E)$ the measure of a set $E \subseteq (a, b)$, Lebesgue is defining $\mathfrak{m}^*(E)$ to be the infimum of the set of all sums of the form $\sum_i \ell(I_i)$ such that $E \subseteq \bigcup_i I_i$ where $I_i = (a_i, b_i]$ and $\ell(I_i) = b_i - a_i$. It's true that Lebesgue doesn't specify the types of intervals, but it doesn't matter what types of intervals you choose to cover E with (I chose left-half open ones because of my upbringing).

[4] Translator's footnote: Lebesgue is defining E to be measurable if $\mathfrak{m}^*(E) + \mathfrak{m}^*((a, b) \cap E^c) = b - a$.

[5] Translator's footnote: Lebesgue is saying that if the E_i are measurable, then $\bigcup_i E_i$ is measurable, if the E_i are pairwise disjoint, then $\mathfrak{m}^*(\bigcup_i E_i) = \sum_i \mathfrak{m}^*(E_i)$, and finally, that $\bigcap_i E_i$ is measurable. The complement of a measurable set is, almost by definition, measurable; moreover, it's not difficult to see that the empty set is measurable. Thus, the collection of measurable sets contains the empty set and is closed under complements and countable unions; later when we define σ-algebras, think about Lebesgue.

[6] Translator's footnote: Lower and upper integrals in the sense of Darboux.

The collection of summable functions obviously contains $y = k$ and $y = x$; therefore, according to (1), it contains all the polynomials and, according to (2), it contains all its limits, therefore it contains all the continuous functions, that is to say, the functions of first class (see Baire, *Annali di Matematica*, 1899), it contains all those of second class, etc. In particular, any derivative bounded in absolute value, being of first class, is summable, and one can show that its integral, considered as function of its upper limit, is an antiderivative. Here is a geometrical application: if $|f'|$, $|\varphi'|$, $|\psi'|$ are bounded, the curve $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$, has a length given by the integral of $\sqrt{(f'^2 + \varphi'^2 + \psi'^2)}$. If $\varphi = \psi = 0$, one obtains the total variation of the function f of bounded variation. If f', φ', ψ' do not exist, one can obtain an almost identical theorem by replacing the derivatives by the Dini derivatives.

Footnotes:

(1) These two conditions imposed *a priori* on any generalization of the integral are obviously compatible, because any integrable derivative, in the sense of Riemann, has as an integral one of its antiderivatives.

(2) If one adds to this collection suitably selected sets of measure zero, one obtains the measurable sets in the sense of Mr. Borel (*Leçons sur la théorie des fonctions*).

Some remarks on Lebesgue's paper

In Section 1.1 of Chapter 1 we shall take a closer look at Lebesgue's theory of integration as he explained in his paper. Right now we shall discuss some aspects he brings up in his paper involving certain defects in the Riemann theory of the integral and how his theory fixes these defects.

The antiderivative problem. One of the fundamental theorems of calculus (FTC) learned in elementary calculus says that for a bounded[7] function $f : [a, b] \to \mathbb{R}$, we have

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{0.1}$$

where F is an antiderivative of f, which means $F'(x) = f(x)$ for all $x \in [a, b]$. It may be hard to accept at first, because it's not stated in a first course in calculus, but the FTC may fail if the integral in (0.1) is the Riemann integral! In fact, there are bounded functions f that are *not* Riemann integrable, but have antiderivatives, thus for such functions the left-hand side of (0.1) does not make sense. In Section ? we shall define such a function due to Vito Volterra (1860–1940) that he published in 1881. With this background, we can understand Lebesgue's inaugural words of his paper:

> In the case of continuous functions, the notions of the integral and antiderivatives are identical. Riemann defined the integral of certain discontinuous functions, but all derivatives are not integrable in the sense of Riemann. Research into the problem

[7]The Riemann integral is only defined for bounded functions, which is why we make this assumption. We would deal with unbounded functions, but then we'll have to discuss improper integrals, which we don't want to get into.

> of antiderivatives is thus not solved by integration, and one can desire a definition of the integral including as a particular case that of Riemann and allowing one to solve the problem of antiderivatives.

In Lebesgue's theory of integral, we shall see that the Fundamental Theorem of Calculus *always* holds for any bounded function with an antiderivative. In this sense, Lebesgue's theory of integral solves the "problem of antiderivatives".

The limit problem. Suppose that for each $n = 1, 2, 3, \ldots$ we are given a function $f_n : [a, b] \to \mathbb{R}$, all bounded by some fixed constant.[8] Also suppose that for each $x \in [a, b]$, $\lim_{n\to\infty} f_n(x)$ exists; since this limit depends on x, the value of the limit defines a function $f : [a, b] \to \mathbb{R}$ such that for each $x \in [a, b]$,

$$f(x) = \lim_{n\to\infty} f_n(x).$$

The function f is bounded since we assumed all the f_n's were bounded by some fixed constant. A question that you've probably seen before in Elementary Real Analysis is the following: Given that the f_n's are Riemann integrable, is it true that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx? \tag{0.2}$$

We shall call this question the "limit problem", which by using the definition of $f(x)$, we can rephrase as follows: Is it true that

$$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx,$$

which is to say, can we switch limits with integrals? In the Riemann integration world, the answer to this question is "No" for the following reason: Even though each f_n is Riemann integrable, it's not necessarily the case that the limit function f is Riemann integrable. Thus, even though the numbers $\int_a^b f_n(x)\,dx$ on the right-hand side of (0.2) may be perfectly well-defined, the symbol $\int_a^b f(x)\,dx$ on the left-hand side of (0.2) may not be defined!

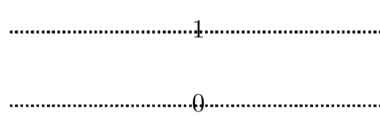
For an example of such a case, we go back to the example Lebesgue brought up in the second-to-last paragraph of his paper where he wrote

> *Example:* The function equal to 0 if x is irrational, equal to 1 if x is rational.

Denoting this function by $f : \mathbb{R} \to \mathbb{R}$, we have

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is called **Dirichlet's function** after Johann Peter Gustav Lejeune Dirichlet (1805–1859) who introduced it in 1829; here's a rough picture of Dirichlet's function:

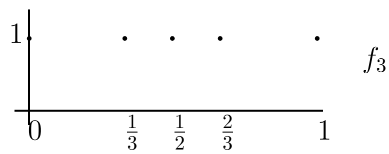


[8]That is, there is a constant C such that $|f_n(x)| \le C$ for all $x \in [a, b]$ and for all n.

It's easy to show that $f : \mathbb{R} \to \mathbb{R}$ is *not* Riemann integrable on any interval $[a, b]$ with $a < b$ (See Exercise 1). Now, in 1898, René-Louis Baire (1874–1932) introduced the following sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, 3, \ldots$, defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x = p/q \text{ is rational in lowest terms with } q \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a picture of f_3 focusing on $x \in [0, 1]$:



Notice that $f_3(x) = 1$ when $x = 0, 1/3, 1/2, 2/3/1$, the rationals with denominators not greater than 3 when written in lowest terms, otherwise $f_3(x) = 0$. More generally, f_n is equal to the zero function except at finitely many points, namely at 0/1, 1/1, 1/2, 1/3, 2/3, . . ., $(n-1)/n$ and 1/1. In particular, f_n is Riemann integrable and for any $a < b$,

$$\int_a^b f_n(x)\, dx = 0;$$

here we recall that the Riemann integral is immune to changes in functions at finitely many points, so as the f_n's differ from the zero function at only finitely many points, $\int_a^b f_n(x)\, dx = \int_a^b 0\, dx = 0$. Also notice that

$$\lim_{n\to\infty} f_n = \text{the Dirichlet function},$$

which as we mentioned earlier is not Riemann integrable. Hence, for this simple example, the limit equality (0.2) is nonsense because the left-hand side of the equality is not defined.

In Lebesgue's theory of integration, we shall see that the limit function f will always be Lebesgue integrable (which Lebesgue mentions in point (2) at the end of the second-to-last paragraph of his paper) and moreover, the equality (0.2) *always* holds when the sequence f_n is bounded. In this sense, Lebesgue's theory of integral gives a positive answer to the "limit problem". Finally, let's discuss

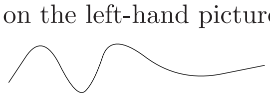
The arc length problem. In the last paragraph of Lebesgue's paper he mentions the following geometric application:

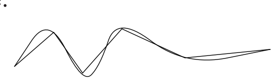
> Here is a geometrical application: if $|f'|$, $|\varphi'|$, $|\psi'|$ are bounded, the curve $x = f(t)$, $y = \varphi(t)$, $z = \psi(t)$, has a length given by the integral of $\sqrt{(f'^2 + \varphi'^2 + \psi'^2)}$.

To elaborate more on this, suppose we are given a curve C in 3-space defined by parametric equations

$$C \;:\; x = f(t)\;,\quad y = \varphi(t)\;,\quad z = \psi(t)\;,\quad a \leq t \leq b,$$

such as shown on the left-hand picture here:





To define L, the length of C, we approximate the curve by a piecewise linear curve, an example of which is shown on the right, and find the length of the approximating curve. Taking closer and closer approximations to the curve by piecewise linear curves, we define the length of the curve L by

$$L := \text{the limit of the lengths of the piecewise linear approximations,} \tag{0.3}$$

provided that the lengths of the piecewise linear approximation approach a specific value. In elementary calculus we learned another formula for the length of the curve:

$$L = \int_a^b \sqrt{(f'(t))^2 + (\varphi'(t))^2 + (\psi'(t))^2}\, dt, \tag{0.4}$$

assuming that the derivatives are bounded. A natural question is: Are the two notions of length, defined by (0.3) and (0.4), equivalent? The answer is "No" if the Riemann integral is used in (0.4)! More precisely, there are curves which have length in the sense of (0.3) but such that $\sqrt{(f'(t))^2 + (\varphi'(t))^2 + (\psi'(t))^2}$ is *not* Riemann integrable; thus, (0.4) is nonsense if the integral is understood in the Riemann sense.

In Lebesgue's theory of integral, we shall see that the two notions of arc length are equivalent. Thus, Lebesgue's theory of integral solves the "arc length problem". There are many other defects in Riemann's integral that Lebesgue's integral fixes, and we'll review and discuss new defects as we progress through the book (for example, see the discussion on multi-dimensional integrals in Chapter ?).

Summary. If we insist on using the Riemann integral, we have to worry about important formulas that are true some of the time; however, using the Lebesgue integral, these "defective formulas" become, for all intents and purposes, correct all of the time. Thus, we can say that

Lebesgue's integral simplifies life!

► **Exercises 0.1.**

1. Using your favorite definition of the Riemann integral you learned in an elementary course on Real Analysis (for instance, via Riemann sums or Darboux sums), prove that Dirichlet's function is not Riemann integrable on any interval $[a, b]$ where $a < b$.