Löse $Ax = b$, $A \in K^{n \times n}$ regulär.

$\Rightarrow \exists A^{-1}$ mit $A^{-1}A = AA^{-1} = I_n$

$\Leftrightarrow$ $Ax = b \Rightarrow \underbrace{A^{-1}A}_{I_n} x = A^{-1}b \Leftrightarrow$

$$\boxed{x = A^{-1}b}$$

## Berechnung einer Inverse von $A \in K^{n \times n}$, A regulär!

$AA^{-1} = I_n \Leftrightarrow$

$$A \begin{pmatrix} | & | & & | \\ s_1 & s_2 & \cdots & s_n \\ | & | & & | \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & 0 & & 0 \\ 0 & \vdots & & 1 \end{pmatrix} \\ e_1 \; e_2 \; \cdots \; e_n \end{array} \Leftrightarrow$$

$$\begin{pmatrix} | & | & & | \\ As_1 & As_2 & \cdots & As_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix} \Leftrightarrow$$

$$\boxed{As_i = e_i \;\; \forall i = 1, \ldots, n}$$

**Ziel:** Bringe $(A \,\vdots\, E_n)$ auf die Gestalt $(I \,\vdots\, A^{-1})$!

D.h.
$$\left(\begin{array}{c|cccc} A & | & | & & | \\ & e_1 & e_2 & \dots & e_n \\ & | & | & & | \end{array}\right) \Rightarrow \left(\begin{array}{c|cccc} I & | & | & & | \\ & s_1 & s_2 & \dots & s_n \\ & | & | & & | \end{array}\right)$$

**Beispiel 2.23**

$$\begin{array}{c} \begin{matrix} 2 & 4 & 0 & 2 & 0 & 0 \end{matrix} \\ \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{l} z_2 - 2z_1 \\ z_3 - z_1 \end{array} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & 3 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad z_2\left(-\tfrac{1}{3}\right) \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & \tfrac{2}{3} & -\tfrac{1}{3} & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad z_3 - z_2 \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 2 & -\frac{5}{3} & \frac{1}{3} & 1 \end{array}\right) z_3\left(\tfrac{1}{2}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{5}{6} & \frac{1}{6} & \frac{1}{2} \end{array}\right) z_2 + z_3$$

$r = 3$, $\mathrm{Rang}(A) = 3 = n \Rightarrow A$ ist regulär.

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0^{0} & 1^{-2} & 0^{0} & -\frac{1}{6}^{\frac{4}{3}} & -\frac{1}{6}^{\frac{1}{3}} & \frac{1}{2}^{-1} \\ 0 & 0 & 1 & -\frac{5}{6} & \frac{1}{6} & \frac{1}{2} \end{array}\right) \begin{array}{l} z_1 + \\ z_2(-2) \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{4}{3} & \frac{1}{3} & -1 \\ 0 & 1 & 0 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{5}{6} & \frac{1}{6} & \frac{1}{2} \end{array}\right)$$

$$\underbrace{\begin{array}{ccc} \frac{4}{3} & \frac{1}{3} & -1 \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \\ -\frac{5}{6} & \frac{1}{6} & \frac{1}{2} \end{array}}_{= A^{-1}}$$

$$\Leftrightarrow \boxed{A^{-1} = \frac{1}{6}\begin{pmatrix} 8 & 2 & -6 \\ -1 & -1 & 3 \\ -5 & 1 & 3 \end{pmatrix}}$$

$$AA^{-1} = A^{-1}A = I_3$$

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Beispiel 3.4 $\quad \varphi : \mathbb{R}^2 = V \longrightarrow \mathbb{R}^2 = W$

$$\varphi(x) = \varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_1 - \frac{1}{2}x_2 \\ 0 \end{pmatrix}.$$

Bild

$$\varphi(x) = s\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ s \in \mathbb{R} \Leftrightarrow x_1\text{... Achse}$$

$$B(\varphi) = \{ y \in \underbrace{\mathbb{R}^2}_{W},\ y = s\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ s \in \mathbb{R} \}$$

Kern

$$\varphi(x) = 0 \Leftrightarrow \begin{pmatrix} x_1 - \frac{1}{2}x_2 \\ 0 \end{pmatrix} = 0 \Leftrightarrow$$

$$\underline{\underline{x_1 = \frac{1}{2}x_2}}$$

$$x = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix},\ x_2 \in \mathbb{R}$$

Eine Gerade durch den Nullp. im $\mathbb{R}^2$

$$\operatorname{Kern}(\varphi) = \Big\{ x \in \underbrace{\mathbb{R}^2}_{V},\ x = s\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix},\ s \in \mathbb{R} \Big\}$$

## Beispiel 3.6

$$A : \mathbb{R}^4 \longrightarrow \mathbb{R}^4,\quad \boxed{A} = \begin{array}{c} \begin{matrix} {\scriptstyle 3} & {\scriptstyle 3} & {\scriptstyle 0} & {\scriptstyle 3} \\ {\scriptstyle 2} & {\scriptstyle 2} & {\scriptstyle 0} & {\scriptstyle 2} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 2 & 3 & 1 & 1 \\ 3 & 4 & 1 & 2 \end{pmatrix} \end{array}$$

**Finde $B(A)$ bzw. die Basis von $B(A)$.**

$$\boxed{A} \xRightarrow[z_4 - 3z_1]{z_3 - 2z_1} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \Rightarrow \begin{matrix} z_3 - z_2 \\ z_4 - z_2 \end{matrix}$$

$$\Rightarrow \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 1 \\ 1 & -1 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} \qquad \operatorname{Rang}(A) = 2 = \operatorname{Dim}(B(A))$$

$$B(A) = \Big\{ y \in \mathbb{R}^4,\ y = s\begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix},\ s,t \in \mathbb{R} \Big\} = \mathcal{L}\Big\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 4 \end{pmatrix} \Big\} \neq \ !$$

$$!\ \neq \Big\{ y \in \mathbb{R}^4,\ y = s\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \\ \end{pmatrix},\ s,t \in \mathbb{R} \Big\}$$

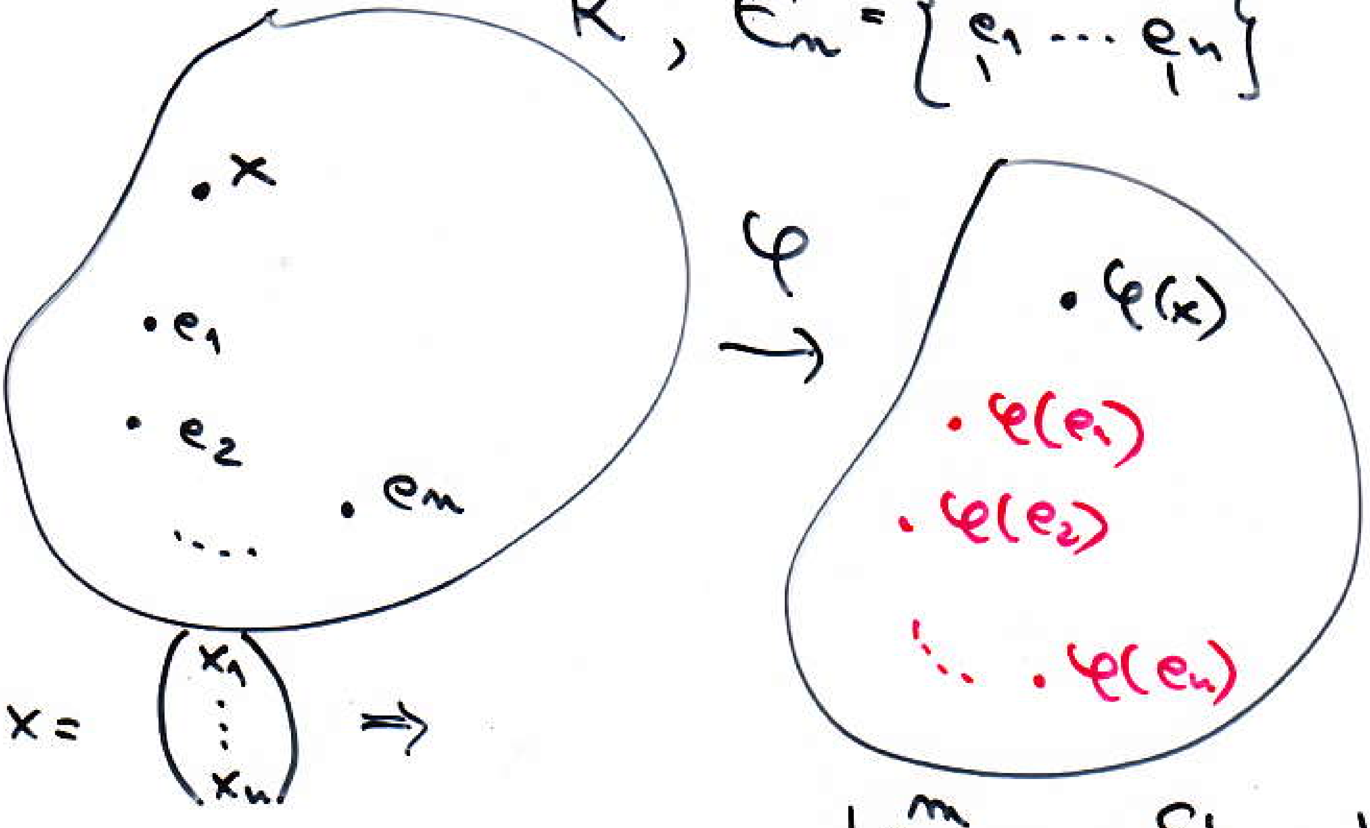
<u>Beispiel</u>: Sei $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \Rightarrow$

$$\mathcal{B}(A) = \left\{ y \in \mathbb{R}^3,\ y = s_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s_3 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$= \mathcal{L}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. = \left\{ y = (s_1 + s_2 + 2 s_3) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \text{ siehe}$$

Es sei $\varphi : V = K^n \longrightarrow W = K^m$ linear.

$K^n$, $E_n = \{ e_1 \dots e_n \}$




Sei $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \Rightarrow$

$K^m$, $E_m = \{ e_1 \dots e_m \}$

$\Rightarrow x = \underline{x_1} e_1 + \underline{x_2} e_2 + \dots + \underline{x_n} e_n$

Wir bezeichnen

$$\varphi(x) = \varphi(x_1 e_1 + x_2 e_2 + \dots + x_n e_n) =$$

$$= \underline{x_1}\, \varphi(e_1) + \underline{x_2}\, \varphi(e_2) + \dots + \underline{x_n}\, \varphi(e_n)$$

$$\overset{!}{=} \begin{pmatrix} | & | & & | \\ \varphi(e_1) & \varphi(e_2) & \dots & \varphi(e_n) \\ | & | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{\varphi(x) = A x \quad \forall x \in K^n !}$$

$$\|$$

$$\begin{pmatrix} | & | & & | \\ \varphi(e_1) & \varphi(e_2) & \dots & \varphi(e_n) \\ | & | & & | \end{pmatrix}$$

D.h. $\varphi$ kann als $A$ dargestellt werden.

Es gilt $\forall x \in K^n$ $\varphi(x) = Ax$ mit

$$A = \begin{pmatrix} | & | & & | \\ \varphi(e_1) & \varphi(e_2) & \dots & \varphi(e_n) \\ | & | & & | \end{pmatrix}$$