Beispiel 3.7

$E_2 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$\varphi : V = \mathbb{R}^2 \longrightarrow W = \mathbb{R}^3$

$E_3 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\varphi(x) = \varphi\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \\ x_2 \end{pmatrix}$$

1. Methode

$$\varphi(x) = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_2 \\ +3x_2 \\ +x_2 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow$$

$$\forall x \in \mathbb{R}^2 \quad \varphi(x) = Ax, \quad A = \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} -1 \\ 3 \\ 1 \end{matrix}} \end{pmatrix} \in \mathbb{R}^{3 \times 2}$$

2. Methode

$$A = \begin{pmatrix} | & | \\ \varphi(e_1) & \varphi(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ 0 & 1 \end{pmatrix}$$

Wir berechnen

- $\varphi(e_1) = \varphi\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \boxed{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}$
- $\varphi(e_2) = \varphi\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \boxed{\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}}$

Allgemeine Basen B in V und C in W:

V, $\operatorname{Dim} V = n$ $\qquad$ W, $\operatorname{Dim} W = m$

$$B = \{b_1, \dots, b_n\} \qquad \xrightarrow{\varphi} \qquad C = \{c_1, c_2, \dots c_m\}$$

$$\bullet\, v \qquad\qquad \bullet\, \varphi(v)$$

Wir berechnen $[v]_B$. Um $[\varphi(v)]_C$ zu berechnen suchen wir eine $m \times n$ Matrix mit

$$[\varphi(v)]_C = [\varphi(B)]_C \, [v]_B.$$

Rechnung:

- $v \in V$, $x := [v]_B \Leftrightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ mit

$$v = x_1 b_1 + x_2 b_2 + \dots + x_n b_n.$$

- $\varphi(v) = \varphi(x_1 b_1 + x_2 b_2 + \dots + x_n b_n) =$

$$\overset{\text{Linearität}}{=} x_1 \varphi(b_1) + x_2 \varphi(b_2) + \dots + x_n \varphi(b_n)$$

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Stelle $\varphi(b_k)$ mit Hilfe der Basis C dar:
$k=1,2,\dots n$

$$\Leftrightarrow \begin{cases} \varphi(b_1) = a_{11}c_1 + a_{21}c_2 + \dots + a_{m1}c_m \\ \varphi(b_2) = a_{12}c_1 + a_{22}c_2 + \dots + a_{m2}c_m \\ \vdots \\ \varphi(b_n) = a_{1n}c_1 + a_{2n}c_2 + \dots + a_{mn}c_m \end{cases}$$

$$\Rightarrow \varphi(v) = x_1(a_{11}c_1 + a_{21}c_2 + \dots + a_{m1}c_m) + x_2(a_{12}c_1 + a_{22}c_2 + \dots + a_{m2}c_m) + \dots + x_n(a_{1n}c_1 + a_{2n}c_2 + \dots + a_{mn}c_m)$$

$$= \boxed{(x_1a_{11} + x_2a_{12} + \dots + x_na_{1n})c_1} + \boxed{(x_1a_{21} + x_2a_{22} + \dots + x_na_{2n})c_2} + \dots + \boxed{(x_1a_{m1} + x_2a_{m2} + \dots + x_na_{mn})c_m}$$

$\Rightarrow$

$$\underline{\underline{[\varphi(v)]_C}} = \begin{pmatrix} x_1 a_{11} + x_2 a_{12} + \dots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \dots + x_n a_{2n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \dots + x_n a_{mn} \end{pmatrix}$$

$$= x_1 \underbrace{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}}_{= [\varphi(b_1)]_C} + x_2 \underbrace{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}}_{= [\varphi(b_2)]_C} + \dots + x_n \underbrace{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}}_{= [\varphi(b_n)]_C}$$

$$= \underbrace{\begin{pmatrix} | & | & & | \\ [\varphi(b_1)]_C & [\varphi(b_2)]_C & \dots & [\varphi(b_n)]_C \\ | & | & & | \end{pmatrix}}_{[\varphi(B)]_C} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\underline{\underline{[v]_B}}}$$

<u>Beispiel 3.8.</u> $V = \mathbb{R}^2$, $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$W = \mathbb{R}^3$, $C = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Definition von $\varphi: \mathbb{R}^2 \to \mathbb{R}^3$:

$$\varphi(b_1) = \varphi\left(\begin{pmatrix}1\\1\end{pmatrix}\right) := \begin{pmatrix}-3\\-2\\2\end{pmatrix} \text{ und}$$

$$\varphi(b_2) = \varphi\left(\begin{pmatrix}0\\1\end{pmatrix}\right) := \begin{pmatrix}4\\4\\5\end{pmatrix}.$$

a) Stelle die Matrix $[\varphi(B)]_C$ auf.

b) Berechne $\varphi(v)$, $v = \begin{pmatrix}-2\\1\end{pmatrix}$.

a)
$$\begin{cases} \begin{pmatrix}-3\\-2\\1\end{pmatrix} = a_{11}\begin{pmatrix}1\\0\\0\end{pmatrix} + a_{21}\begin{pmatrix}1\\2\\3\end{pmatrix} + a_{31}\begin{pmatrix}-1\\0\\1\end{pmatrix} \\ \begin{pmatrix}4\\4\\5\end{pmatrix} = a_{12}\begin{pmatrix}1\\0\\0\end{pmatrix} + a_{22}\begin{pmatrix}1\\2\\3\end{pmatrix} + a_{32}\begin{pmatrix}-1\\0\\1\end{pmatrix} \end{cases}$$

$\Leftrightarrow$

$$\left(\begin{array}{ccc|cc} 1 & 1 & -1 & -3 & 4 \\ 0 & 2 & 0 & -2 & 4 \\ 0 & 3 & 1 & 1 & 5 \end{array}\right) \Rightarrow \text{G-Elim.}$$

$a_{11} = 3, a_{21} = -1, a_{31} = 5$ und

$a_{12} = 1, a_{22} = 2, a_{32} = -1$

$$\Rightarrow [\varphi(b_1)]_C = \begin{pmatrix}3\\-1\\5\end{pmatrix}, \; [\varphi(b_2)]_C = \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\Rightarrow [\varphi(B)]_C = \begin{pmatrix}3 & 1\\-1 & 2\\5 & -1\end{pmatrix}.$$

$$\Rightarrow \left[\varphi(B)\right]_C = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ 5 & -1 \end{pmatrix}$$

b) gegeben ist $v = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

und es gilt $\left[\varphi(v)\right]_C = \left[\varphi(B)\right]_C \left[v\right]_B$.

Zunächst berechne $\left[v\right]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Leftrightarrow$

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow c_1 = -2, \quad c_2 = 1$$

$$\Rightarrow \left[v\right]_B = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Probe: $-2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2+1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \checkmark$

$$\Rightarrow \left[\varphi(v)\right]_C = \begin{pmatrix} 3 & 1 \\ -1 & 2 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ -11 \end{pmatrix}$$

$$\Rightarrow \varphi(v) = -5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 11\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{\varphi(v) = \begin{pmatrix} 10 \\ 8 \\ 1 \end{pmatrix}}$$

Direktes Ausrechnen von $\varphi(v)$:

Für v gilt: $v = -2b_1 + b_2 = -2\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$

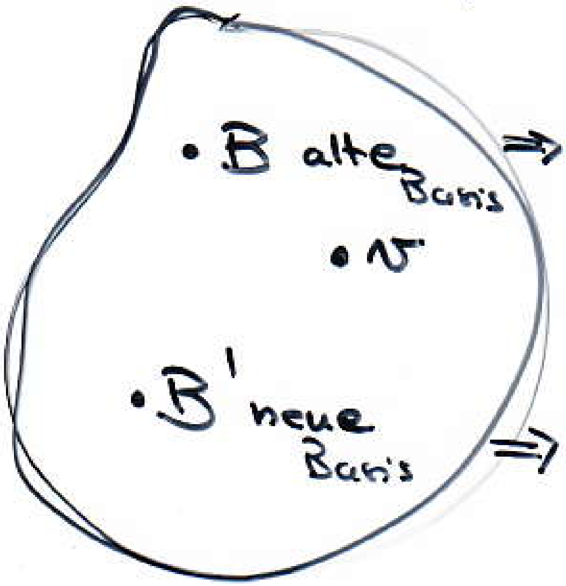
$$\Rightarrow \quad \varphi(v) = \varphi\left(-2\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) \overset{\text{Linearität}}{=} -2\varphi\left(\begin{pmatrix}1\\1\end{pmatrix}\right) + \varphi\left(\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= -2\begin{pmatrix}-3\\-2\\2\end{pmatrix} + \begin{pmatrix}4\\4\\5\end{pmatrix} = \begin{pmatrix}10\\8\\1\end{pmatrix}$$

Basistransf.

(1.) Aufgabe

V, $\dim V = n$



$\Rightarrow [v]_B$

$\Rightarrow [v]_{B'}$

Frage:

Wie lautet die Matrix $T_{B' \leftarrow B}$ mit

$$[v]_{B'} = \underline{T_{B' \leftarrow B}} [v]_B \; ?$$

Definition 3.3.

Mit $T_{B' \leftarrow B}$ aus dieser Definition gilt

$$[v]_{B'} = T_{B' \leftarrow B} [v]_B .$$

Beweis:

Sei

$$[v]_B := \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in K^n, \quad [v]_{B'} := \begin{pmatrix} c_1' \\ c_2' \\ \vdots \\ c_n' \end{pmatrix} \in K^n.$$

$$\Leftrightarrow \begin{cases} v = c_1 b_1 + c_2 b_2 + \dots + c_n b_n \\ v = c_1' b_1' + c_2' b_2' + \dots + c_n' b_n' \end{cases}$$

Wir stellen die Vektoren der alten Basis mit Hilfe der neuen Basis B' dar, und erhalten

$$\begin{cases} b_1 = t_{11} b_1' + t_{21} b_2' + \dots + t_{n1} b_n' \\ b_2 = t_{12} b_1' + t_{22} b_2' + \dots + t_{n2} b_n' \\ \vdots \\ b_n = t_{1n} b_1' + t_{2n} b_2' + \dots + t_{nn} b_n' \end{cases}$$

$$\begin{aligned} v = \; & c_1 (t_{11} b_1' + t_{21} b_2' + \dots + t_{n1} b_n') \\ & + c_2 (t_{12} b_1' + t_{22} b_2' + \dots + t_{n2} b_n') \\ & \vdots \\ & + c_n (t_{1n} b_1' + t_{2n} b_2' + \dots + t_{nn} b_n') \end{aligned}$$

$$\Rightarrow v = \boxed{(c_1 t_{11} + c_2 t_{12} + \dots + c_n t_{1n})\, b_1'}$$

$$+ \boxed{(c_1 t_{21} + c_2 t_{22} + \dots + c_n t_{2n})\, b_2'}$$

$$\vdots$$

$$+ \boxed{(c_1 t_{n1} + c_2 t_{n2} + \dots + c_n t_{nn})\, b_n'}$$

$$\Downarrow$$

$$[v]_{B'} = \begin{pmatrix} c_1 t_{11} + c_2 t_{12} + \dots + c_n t_{1n} \\ c_1 t_{21} + c_2 t_{22} + \dots + c_n t_{2n} \\ \vdots \\ c_1 t_{n1} + c_2 t_{n2} + \dots + c_n t_{nn} \end{pmatrix}$$

$$= c_1 \begin{pmatrix} t_{11} \\ t_{21} \\ \vdots \\ t_{n1} \end{pmatrix} + c_2 \begin{pmatrix} t_{12} \\ t_{22} \\ \vdots \\ t_{n2} \end{pmatrix} + \dots + c_n \begin{pmatrix} t_{1n} \\ t_{2n} \\ \vdots \\ t_{nn} \end{pmatrix}$$

$$= \overbrace{\begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ t_{21} & t_{22} & \dots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix}}^{T_{B' \leftarrow B} :=} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{= [v]_B}$$

vgl. Def 3.3.

Beispiel 3.11.

$V = \mathbb{R}^3$

$B = \left\{ \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} \right\} = \{ b_1, b_2, b_3 \}$

$B' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = \{ b_1', b_2', b_3' \}$

a) $T_{B' \leftarrow B}$?

b) $[v]_B$, $[v]_{B'}$; $v = \begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix}$?

a)
$$\begin{cases} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = t_{11} b_1' + t_{21} b_2' + t_{31} b_3' \\ \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} = t_{12} b_1' + t_{22} b_2' + t_{32} b_3' \\ \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} = t_{13} b_1' + t_{23} b_2' + t_{33} b_3' \end{cases}$$

$\Updownarrow$

Das rechne ich doch! Siehe Nachträge.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & -4 & 6 \\ 1 & 1 & 0 & 3 & 4 & -2 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right) \overset{\cdots}{\Rightarrow} \left(I_3 \mid T_{B' \leftarrow B} \right)$$

$$T_{B'\leftarrow B} = \begin{pmatrix} 5 & 12 & -11 \\ -2 & -8 & 9 \\ 1 & 8 & -8 \end{pmatrix}$$

b)

$$\begin{pmatrix} -2 \\ 7 \\ -1 \end{pmatrix} = c_1 b_1 + c_2 b_2 + c_3 b_3$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} | & | & | & -2 \\ b_1 & b_2 & b_3 & 7 \\ | & | & | & -1 \end{array}\right) \overset{GE}{\underset{\dots}{\Longrightarrow}} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = [v]_B = \underline{\underline{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}}$$

…und auch das!

$$\underline{\underline{[v]_{B'}}} = T_{B'\leftarrow B}[v]_B =$$

$$= \begin{pmatrix} 5 & 12 & -11 \\ -2 & -8 & 9 \\ 1 & 8 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 17 \\ -10 \\ 9 \end{pmatrix}}}.$$

<u>Nachträge:</u>

1.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & -4 & 6 \\ 1 & 1 & 0 & 3 & 4 & -2 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad z_2 - z_1 \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & -4 & 6 \\ 0 & 0 & 1 & 1 & 8 & -8 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right) \quad z_2 \leftrightarrow z_3 \Longrightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & -4 & 6 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 8 & -8 \end{array}\right) \begin{array}{l} z_1 + z_3 \\ z_2 - z_3 \end{array} \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 4 & -2 \\ 0 & 1 & 0 & -2 & -8 & 9 \\ 0 & 0 & 1 & 1 & 8 & -8 \end{array}\right) \; z_1 - z_2 \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 12 & -11 \\ 0 & 1 & 0 & -2 & -8 & 9 \\ 0 & 0 & 1 & 1 & 8 & -8 \end{array}\right)$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{T_{B' \leftarrow B}}$$

Also,

$$T_{B' \leftarrow B} = \begin{pmatrix} 5 & 12 & -11 \\ -2 & -8 & 9 \\ 1 & 8 & -8 \end{pmatrix}$$

2.

$$\left(\begin{array}{ccc|c} 2 & -4 & 6 & -2 \\ 3 & 4 & -2 & 7 \\ -1 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} (-1)\, z_3 \\ z_1 \leftrightarrow z_3 \end{array} \Rightarrow$$

$$\begin{array}{c} \begin{array}{cccc} 3 & 0 & -3 & 3 \end{array} \\ \begin{array}{cccc} 2 & 0 & -2 & 2 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 3 & 4 & -2 & 7 \\ 2 & -4 & 6 & -2 \end{array}\right) \end{array} \begin{array}{l} z_2 - 3z_1 \\ z_3 - 2z_1 \end{array}$$

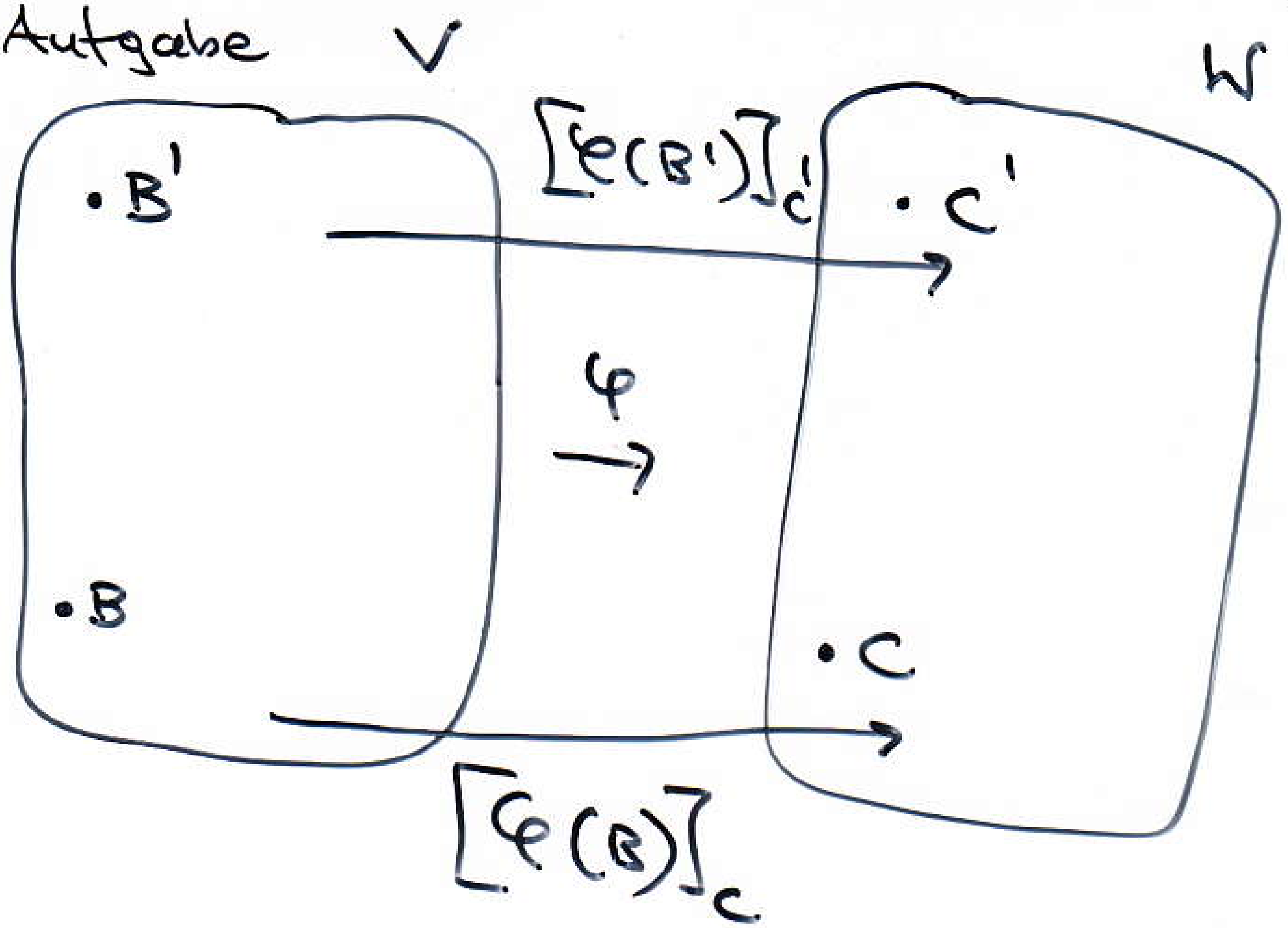
$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 4 & 1 & 4 \\ 0 & -4 & 8 & -4 \end{array}\right) \overset{z_3+z_2}{\Longrightarrow}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 4 & 1 & 4 \\ 0 & 0 & 9 & 0 \end{array}\right) \Longleftrightarrow$$

$$\begin{cases} c_1 - c_3 = 1 \\ 4c_2 + c_3 = 4 \\ 9c_3 = 0 \Rightarrow c_3 = 0 \end{cases} \Rightarrow c_4 = 1 \Rightarrow c_1 = 1$$

$$\Longleftrightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = [v]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

2. Aufgabe



Hat man eine Darstellg von φ mit Hilfe der Matrix $[\varphi(B)]_C$ (bezüglich der Basen B, C), so stellen wir die Frage wie man diese Information nutzen kann, um die Darstellg von φ bezüglich der Bases B', C' herzuleiten, also wie sieht

$$[\varphi(B')]_{C'} \text{ aus?}$$