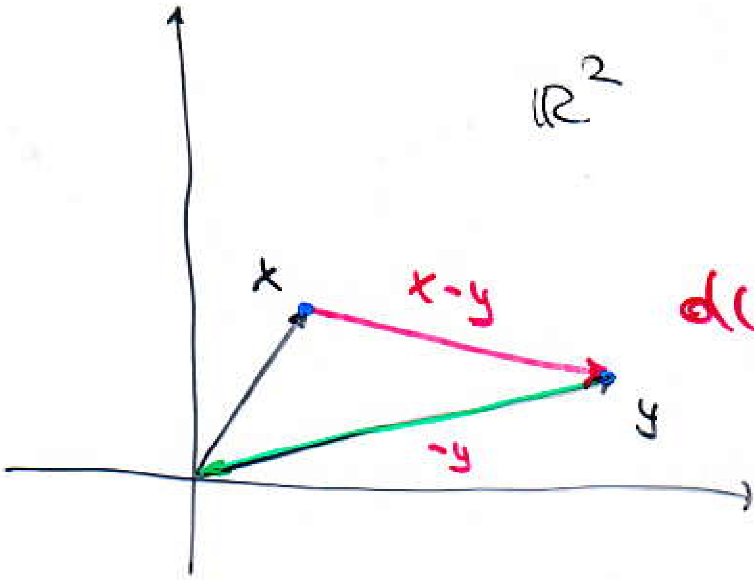
Zum Beispiel 5.5



$d(x,y) := \|x-y\|_2$

Satz 5.3

$x \neq 0,\ y \neq 0;\ \alpha := \langle y,y\rangle,\ \beta := \langle x,y\rangle.$

Betrachte

$$\langle \alpha x-\beta y, \alpha x-\beta y\rangle \geq 0 \iff$$

$$\langle \alpha x, \alpha x-\beta y\rangle - \langle \beta y, \alpha x-\beta y\rangle \geq 0 \iff$$

$$\langle \alpha x, \alpha x\rangle - \langle \cancel{\alpha x}, \beta y\rangle - \left(\langle \beta y, \alpha x\rangle - \langle \beta y, \beta y\rangle\right) \geq 0$$

$$\Updownarrow$$

$$\alpha^2\langle x,x\rangle - \alpha\beta\langle x,y\rangle - \beta\alpha\underbrace{\langle y,x\rangle}_{=\langle x,y\rangle} + \beta^2\langle y,y\rangle \geq 0$$

$$\Updownarrow$$

$$\Rightarrow \alpha^2 \langle x,x\rangle - 2\alpha\beta \langle x,y\rangle + \beta^2 \langle y,y\rangle \geq 0$$

$$\Rightarrow \langle y,y\rangle^2 \langle x,x\rangle - 2\langle y,y\rangle \langle x,y\rangle^2 + \langle x,y\rangle^2 \langle y,y\rangle \geq 0$$

$y \neq 0 \Rightarrow \langle y,y\rangle \neq 0 \Rightarrow$

$$\langle y,y\rangle \langle x,x\rangle \underbrace{- 2\langle x,y\rangle^2 + \langle x,y\rangle^2}_{= -\langle x,y\rangle^2} \geq 0$$

$$\Rightarrow \langle x,y\rangle^2 \leq \langle y,y\rangle \langle x,x\rangle$$

$$\Rightarrow \langle x,y\rangle^2 \leq \|y\|_2^2 \|x\|_2^2 \quad \sqrt{}$$

$$|\langle x,y\rangle| \leq \|y\|_2 \|x\|_2 \quad \square.$$

$$\forall x,y \in V$$

Winkel zwischen x und y:

$$\|x+y\|_2^2 = (x+y, x+y) = (x+y)\cdot(x+y)$$

$$= x(x+y) + y(x+y) = (x,x) + (x,y) + \underbrace{(y,x)}_{=(x,y)}$$

$$+ (y,y) = (x,x) + 2(x,y) + (y,y) =$$

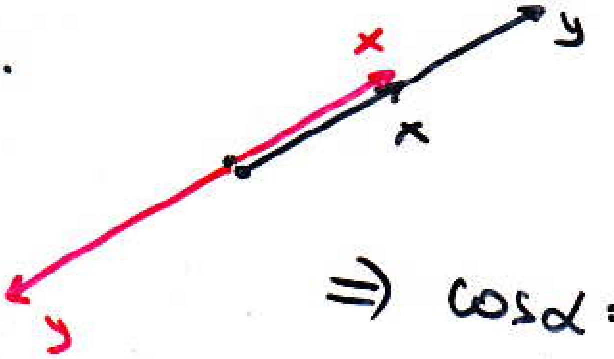
$$= \|x\|_2^2 + 2\,x\cdot y + \|y\|_2^2 .$$

Bemerkung zur Definition 5.9:

$\alpha \in [0,\pi]$; $\cos\alpha := \dfrac{\langle x,y\rangle}{\|x\|_2\,\|y\|_2}$; $x \neq 0$ und $y \neq 0$

1. $$|\cos\alpha| = \left|\frac{\langle x,y\rangle}{\|x\|_2\,\|y\|_2}\right| = \frac{|\langle x,y\rangle| \le \|x\|_2\,\|y\|_2}{\|x\|_2\,\|y\|_2}$$

$$\le \frac{\|x\|_2\,\|y\|_2}{\|x\|_2\,\|y\|_2} = 1 \text{; d.h. } |\cos\alpha| \le 1 .$$

2. 

Sei $x = sy$, $s \in \mathbb{R}\setminus\{0\}$

$$\Rightarrow \cos\alpha = \frac{\langle sy,y\rangle}{\|sy\|_2\,\|y\|_2} = \frac{s\cancel{\|y\|^2}}{|s|\,\cancel{\|y\|_2^2}}$$

$$= \begin{cases} 1 \\ -1 \end{cases} \Rightarrow \alpha = 0 \text{ oder } \alpha = \pi$$

3. $$\cos\alpha = 0 \Leftrightarrow \alpha = \pi/2 \Leftrightarrow \langle x,y\rangle = 0$$

Beispiel S. 9.

$V = \mathbb{R}^3$; $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ $x \cdot y := \sum_{i=1}^{3} x_i y_i$.

$U := \{ x \in \mathbb{R}^3, \ x \perp (1,1,2)^T \} \Rightarrow$

$$\left. \begin{array}{l} x \perp (1,1,2) \Leftrightarrow x \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0 \Leftrightarrow \\ \boxed{x_1 + x_2 + 2x_3 = 0} \end{array} \right\} \Rightarrow$$

$$U = \{ x \in \mathbb{R}^3, \ \underbrace{x_1 + x_2 + 2x_3 = 0}_{\Updownarrow} \}.$$

$x_1 = -x_2 - 2x_3$, $x_2 := s$, $x_3 := t$

$\Rightarrow$ $x_1 = -s - 2t$ $\Leftrightarrow$

$$U = \left\{ x \in \mathbb{R}^3, \ x = \begin{pmatrix} -s-2t \\ s \\ t \end{pmatrix} \right\} =$$

$$= \left\{ x \in \mathbb{R}^3, \ x = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \ s,t \in \mathbb{R} \right\}$$

Aufgabe: Gebe eine ONB für $\underset{an}{U}$!

Notation:

$OGB = \{ u_1, u_2 \}$, $ONB = \left\{ \frac{u_1}{\|u_1\|_2}, \frac{u_2}{\|u\|_2} \right\}$

1. $u_1 := \begin{pmatrix}-1\\1\\0\end{pmatrix}$

2. $$\left.\begin{array}{l}\text{Suche } u_2 \in U \text{ mit } u_1 \cdot u_2 = \emptyset \\ u_2 \in U \Leftrightarrow u_2 = s\begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\0\\1\end{pmatrix}\end{array}\right\} \Rightarrow$$

$$u_1 \perp u_2 \Leftrightarrow \begin{pmatrix}-1\\1\\0\end{pmatrix} \cdot \left(s\begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\0\\1\end{pmatrix}\right) = \emptyset$$

$$\Leftrightarrow \begin{pmatrix}-1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-s-2t\\s\\t\end{pmatrix} = \emptyset$$

$$\Leftrightarrow s + 2t + s = \emptyset \Leftrightarrow 2t + 2s = \emptyset$$

$$\Leftrightarrow \underline{s = -t} \Rightarrow u_2 = -t\begin{pmatrix}-1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\0\\1\end{pmatrix} = t\begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

Wähle $u_2 = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$ und prüfe, ob $u_1 \perp u_2$

: $$\begin{pmatrix}-1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-1\\-1\\1\end{pmatrix} = 1 - 1 = \emptyset \quad ✓$$

$$OGB = \{u_1, u_2\} = \left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$$

$$\|u_1\|_2 = \sqrt{1+1} = \sqrt{2}, \quad \|u_2\|_2 = \sqrt{1+1+1} = \sqrt{3}$$

$$\text{ONB} = \left\{ \frac{u_1}{\|u_1\|_2}, \frac{u_2}{\|u_2\|_2} \right\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Satz 5.8

$$B = \{ b_1, b_2, \dots, b_n \} \text{ ONB} \quad \begin{cases} b_i \perp b_j & i \neq j \\ \|b_i\|_2 = 1 & \forall i \end{cases}$$

$$x = x_1 b_1 + x_2 b_2 + \dots + x_n b_n \iff [x]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Fourierkoeff.

1) $$x_i = \langle x, b_i \rangle \quad \forall i$$

Beweis: i fest; $1 \leq i \leq n$

$$\langle x, b_i \rangle = \langle x_1 b_1 + x_2 b_2 + \dots + x_{i-1} b_{i-1} + x_i b_i + x_{i+1} b_{i+1} + \dots + x_n b_n, b_i \rangle$$

$$= \overbrace{\langle x_1 b_1, b_i \rangle}^{x_1 \overbrace{\langle b_1, b_i \rangle}^{=0}} + \overbrace{\langle x_2 b_2, b_i \rangle}^{x_2 \overbrace{\langle b_2, b_i \rangle}^{=0}} + \dots + \overbrace{\langle x_{i-1} b_{i-1}, b_i \rangle}^{x_{i-1} \overbrace{\langle b_{i-1}, b_i \rangle}^{=0}} + \underbrace{\langle x_i b_i, b_i \rangle}_{x_i \underbrace{\langle b_i, b_i \rangle}_{=1}} + \overbrace{\langle x_{i+1} b_{i+1}, b_i \rangle}^{x_{i+1} \overbrace{\langle b_{i+1}, b_i \rangle}^{=0}} + \dots + \underbrace{\langle x_n b_n, b_i \rangle}_{x_n \underbrace{\langle b_n, b_i \rangle}_{=0}} = x_i \quad \forall i$$

2) $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = [x]_B \cdot [y]_B$

Notation

$$x = \sum_{j=1}^{n} x_j b_j, \quad y = \sum_{k=1}^{n} y_k b_k$$

d.h. $[x]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad [y]_B = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

Beweis:

$$\langle x, y \rangle = \left\langle \sum_j x_j b_j, \sum_k y_k b_k \right\rangle =$$

$$= \langle x_1 b_1 + x_2 b_2 + \dots + x_n b_n, y_1 b_1 + y_2 b_2 + \dots + y_n b_n \rangle$$

$$= \langle x_1 b_1, y_1 b_1 + y_2 b_2 + \dots + y_n b_n \rangle +$$
$$\langle x_2 b_2, y_1 b_1 + y_2 b_2 + \dots + y_n b_n \rangle +$$
$$\vdots$$
$$\langle x_n b_n, y_1 b_1 + y_2 b_2 + \dots + y_n b_n \rangle =$$

$$= \langle x_1 b_1, y_1 b_1 \rangle + \langle x_2 b_2, y_2 b_2 \rangle + \dots \langle x_n b_n, y_n b_n \rangle$$
$$= x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i \qquad \square$$