Beispiel 6.3

$$\dot{x}(t) = A x(t) \quad , A \in \mathbb{R}^{n \times n}$$

Gesucht $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$, $x_i : [a,b] \to \mathbb{R}$.

$n = 1$ $\quad \dot{x}(t) = S x(t)$

1) $x(t) = c e^{Sx}$, $c \in \mathbb{R}$

Probe: $\underline{\dot{x}(t)} = c \cdot S e^{Sx} = S(c e^{Sx}) = \underline{S x(t)}$

2) $x(t) = e^{Sx} + c$ ... keine Lösung wegen

$\dot{x}(t) = S e^{Sx}$

$\lightning$ $S e^{Sx} = S(e^{Sx} + c) = S e^{Sx} + Sc$

Beispiel: $n = 2$

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \Leftrightarrow$$

$$\begin{cases} \dot{x}_1(t) = x_1(t) - x_2(t) \\ \dot{x}_2(t) = 5 x_1(t) + 7 x_2(t) \end{cases}$$

2

Es sei $\dot{x} = Ax$, $x = x(t)$

Ansatz $x(t) := e^{\alpha t} v$; $\alpha = ?$ $v = ?$

$\Rightarrow \dot{x}(t) = \alpha e^{\alpha t} v \Leftrightarrow$

$$\alpha \cancel{e^{\alpha t}} v = A \cancel{e^{\alpha t}} v \quad \Big| \cdot \frac{1}{e^{\alpha t}}$$

$$\boxed{\alpha v = A v} \quad \text{EWP}$$

α ... EW und $v \neq 0$ EV.

Beispiel 6.4

$$B\ddot{x}(t) + D\dot{x}(t) + Cx(t) = f(t)$$

$x(t) = (x_1, x_2, \ldots, x_n)(t)$

Spezialfall: $D = \emptyset$, $f(t) \equiv \emptyset$

$$\boxed{B\ddot{x}(t) + Cx(t) = \emptyset}$$

B, C ... symm. und pos. definit

<u>Ansatz</u> $x(t) := (\alpha \cos\mu t + \beta \sin\mu t)v \neq 0$

$$\begin{cases} \alpha, \beta, \mu \in \mathbb{R} \\ v \in \mathbb{R}^n \end{cases}$$

$$\Rightarrow \quad \dot{x}(t) = (-\alpha\mu \sin\mu t + \beta\mu \cos\mu t)v$$

$$\Rightarrow \quad \ddot{x}(t) = (-\alpha\mu^2 \cos\mu t - \beta\mu^2 \sin\mu t)v$$

$$= -\mu^2(\alpha \cos\mu t + \beta \sin\mu t)v$$

$$= -\mu^2 x(t)$$

$$\Rightarrow \quad B\ddot{x}(t) + Cx(t) = 0 \quad \Leftrightarrow$$

$$B\Big(-\mu^2 \underbrace{(\alpha \cos\mu t + \beta \sin\mu t)v}_{\neq 0}\Big) + C\Big(\underbrace{(\alpha \cos\mu t + \beta \sin\mu t)v}_{\neq 0}\Big) = 0$$

$$\Rightarrow \quad \big(B(-\mu^2) + C\big)v = 0$$

$B \ldots$ pos. def $\Rightarrow$ B regulär $\Rightarrow$

$$B^{-1}\left(B(-\mu^2) + C\right)v = \emptyset \quad \Rightarrow$$

$$\left(-\mu^2 I + B^{-1}C\right)v = \emptyset \Rightarrow$$

$$\left.\begin{array}{l} \lambda := \mu^2 \\ B^{-1}C =: A \end{array}\right\} \Rightarrow \left(-\lambda I + A\right)v = 0$$

$$\Longleftrightarrow \boxed{Av = \lambda v}$$

Lösung des EWPs: $A \in \mathbb{R}^{n \times n}$

$$\Longleftrightarrow Av = \lambda v, \quad v \neq 0.$$

$$\Longleftrightarrow Av - \lambda I v = \emptyset$$

$$(*) \quad (A - \lambda I)v = \emptyset, \quad v \neq 0$$

$Bv = \emptyset \Longleftrightarrow v = 0$
Rang $B = n =$ voll

$\Updownarrow$

Damit $(*)$ eine Lösung $v \neq 0$ hat muß $A - \lambda I$ singulär sein! $\Longleftrightarrow$ $\boxed{\mathrm{Det}(A - \lambda I) = \emptyset}$

Beispiel 6.5. $A \in \mathbb{R}^{n \times n}$, $n=2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Berechne die EW λ und die EV $v \neq 0$.

1. Schritt: EW: $\operatorname{Det}(A - \lambda I) =$

$$= \operatorname{Det}\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) =$$

$$= \operatorname{Det}\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} =$$

$$= (2-\lambda)^2 - 1 = (2-\lambda-1)(2-\lambda+1) =$$

$$= (-\lambda+1)(-\lambda+3) = \emptyset$$

$$\Leftrightarrow \quad \lambda_1 = 1 \qquad \lambda_2 = 3$$

2. Schritt: EVen

$\underline{\lambda_1 = 1}$ $\quad (A - \lambda I)v = (A - I)v =$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v = \emptyset$$

$\operatorname{Rang}(A - \lambda I) = 1 \Rightarrow \operatorname{Dim}\operatorname{Kern}(A - I)$

$$= \operatorname{Dim} E_1(\lambda = 1) = 1$$

$\Rightarrow \quad v_1 + v_2 = \emptyset \Rightarrow v_1 = -v_2$

Wähle $v_2 := s \Rightarrow v_1 = -s \Rightarrow$

$$v = \begin{pmatrix} -s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad s \neq 0$$

$$\Rightarrow \quad \underline{\underline{E_1(\lambda = 1) = \mathcal{L}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}}}$$

$\underline{\lambda_2 = 3}$ $\quad (A - 3I)v = 0 \Leftrightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} v = \emptyset$

$\operatorname{Rang}(A - 3I) = 1 \Rightarrow \underbrace{\operatorname{Dim} \operatorname{Kern}}_{(A-3I)} = 1$

$$\Rightarrow \quad \underline{\underline{\dim E(\lambda = 3) = 1}}$$

$\Leftrightarrow \quad -v_1 + v_2 = \emptyset \Rightarrow +v_1 = +v_2 \Leftrightarrow$

Wähle $v_2 = s \Rightarrow v_1 = s \Rightarrow$

$$v = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s \neq 0.$$

$$\Rightarrow \quad E_2(\lambda = 3) = \mathcal{L}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Beispiel 6.6.

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

$$\mathrm{Det}(A - \lambda I) = \mathrm{Det} \begin{pmatrix} -\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ 2 & 1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)\left(-\lambda(1-\lambda)+1\right) =$$

$$= (3-\lambda)\underbrace{\left(\lambda^2 - \lambda + 1\right)}\ = 0 \iff$$

$$\underline{\lambda_1 = 3} \quad \text{oder } \lambda_{2,3} = \underline{\frac{1 \pm \sqrt{1-4}}{2}}$$

$$\Rightarrow \underline{\lambda_1 = 3}, \quad \underline{\lambda_2 = \frac{1+\sqrt{3}i}{2}}, \quad \underline{\lambda_3 = \frac{1-\sqrt{3}i}{2}}$$

$$\boxed{p(\lambda) = -\lambda^3 + 4\lambda^2 - 4\lambda + 3}$$

$$= c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$$

$$c_3 = (-1)^3 = -1$$

Vgl. Satz 6.2.

$$c_2 = (-1)^2(0 + 1 + 3) = 4$$

$$c_0 = \mathrm{Det}(A) = -(-3) = 3$$

Beispiel 6.7

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \mathrm{Det}(A - \lambda I) =$$

$$= \mathrm{Det} \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(1-\lambda)(2-\lambda) = \emptyset$$

$$\left.\begin{array}{l} \Rightarrow \lambda_1 = 1, \ \underline{m_1 = 2}, \ g_1 = ? \\ \Rightarrow \lambda_2 = 2, \ m_2 = 1, \ g_2 = ? \end{array}\right\}$$

$\underline{\underline{\lambda_1 = 1}}$ $\quad (A - \lambda_1 I)v = 0 \Leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} v = \emptyset$

$\Rightarrow$ Rang $(A - I) = 2 \Rightarrow$ Dim Kern $(A - I) = 1$

$\Rightarrow$ Dim $E_1(\lambda = 1) = 1 \Rightarrow \underline{\underline{g_1 = 1}}$

$\Rightarrow v_2 = 0, \ v_3 = 0, \ v_1 = s \Rightarrow v = s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, s \neq 0$

$\Rightarrow \quad E_1(\lambda=1) = \mathcal{L}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}.$

$(A-\lambda_2 I)v = \emptyset$

$\underline{\lambda_2 = 2} \iff (A-2I)v = \emptyset \iff \begin{pmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} v = \emptyset$

$\text{Rang}(A-2I) = 2 \Rightarrow \text{Dim Kern}(A-2I) = 1$

$\Rightarrow \text{Dim } E_2(\lambda_2=2) = 1 \Rightarrow \underline{\underline{g_2 = 1}}$

$\Rightarrow \quad v_2 = \emptyset, \ v_1 = \emptyset \quad v_3 = s \Rightarrow v = s\begin{pmatrix}0\\0\\1\end{pmatrix}$

$s \neq 0$

$\Downarrow$

$E_2(\lambda_2=2) = \mathcal{L}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$

<u>Bemerkung zu Def. 6.6.</u>

$\left\{\begin{array}{l} A \in K^{n\times n} \\ \lambda_1, \lambda_2, \dots, \lambda_n \\ v_1, v_2, \dots, v_n \end{array}\right. \iff n_i = g_i \ \forall i$

Es gilt $Av_i = \lambda_i v_i, \ i = 1,\dots,n \iff$

$$\begin{pmatrix} | & | & & | \\ Av_1 & Av_2 & \dots & Av_n \\ | & | & & | \end{pmatrix} =$$

$$= \begin{pmatrix} | & | & & | \\ \lambda_1 v_1 & \lambda_2 v_2 & \dots & \lambda_n v_n \\ | & | & & | \end{pmatrix}$$

$\Leftrightarrow$ Linearität von A.

$$A \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_n \\ | & | & & | \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \lambda_n \end{pmatrix}}_{D}$$

$$\Leftrightarrow \boxed{A = X D X^{-1}}$$

Beispiel:

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -3 & -2 & 3 \\ -2 & -2 & 3 \end{pmatrix}$$

1. Schritt EW: $\overbrace{\mathrm{Det}(A - \lambda I)}^{p(\lambda) :=} =$

$$= -\lambda^3 + \lambda^2 + \lambda - 1 = \varnothing$$

$\lambda_1 = 1$, wegen $p(1) = -1 + 1 + 1 - 1 = \varnothing$

$$-\lambda^3+\lambda^2+\lambda-1 : (\lambda-1) = -\lambda^2+1$$
$$\underline{\mp\lambda^3 \pm \lambda^2}$$
$$= \quad = \quad \lambda - 1$$
$$\underline{-\lambda \mp 1}$$
$$= \quad =$$

$$\Rightarrow p(\lambda) = -(\lambda-1)(\lambda^2-1) \Leftrightarrow$$

$$\lambda_1 = 1 \quad \lambda_{2,3} = \pm 1$$

$$\Leftrightarrow \boxed{p(\lambda) = -(\lambda-1)^2(\lambda+1)}$$

$$\begin{cases} \lambda_1 = 1 \text{ mit } n_1 = 2, & g_1 = ? \\ \lambda_2 = -1 \text{ mit } n_2 = 1, & g_2 = 1 \end{cases}$$

<u>2. Schritt</u> EV:

<u>$\lambda_1 = 1$</u> $\quad (A-I)v = \begin{pmatrix} -1 & -1 & 1 \\ -3 & -3 & 3 \\ -2 & -2 & 2 \end{pmatrix} v = \emptyset$

$\Rightarrow \text{Rang}(A-I) = 1 \Rightarrow \text{Dim Kern}(A-I) = 2$
$\Leftrightarrow g_1 = 2$

$\Rightarrow -v_1 - v_2 + v_3 = \emptyset \Rightarrow v_3 = v_1 + v_2$

Wähle $v_1 := s,\ v_2 := t \Rightarrow v_3 = s+t$

$$\Rightarrow v = \begin{pmatrix} s \\ t \\ s+t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\Leftrightarrow \text{Kern}(A-I) = E_1(\lambda=1) = \mathcal{L}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$\underline{\lambda_2 = -1}$

$$\begin{array}{ccc} 2 & -2 & 2 \\ 3 & -3 & 3 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ -3 & -1 & 3 \\ -2 & -2 & 4 \end{pmatrix} v = \emptyset \Rightarrow$$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ \emptyset & -4 & 6 \\ \cancel{\emptyset} & \cancel{-4} & \cancel{6} \end{pmatrix} v = \emptyset$$

$$\text{Rang}(A+I) = 2 \Rightarrow \text{Dim}\,\overline{\text{Kern}}(A+I) = 1$$

$$\begin{cases} v_1 - v_2 = -v_3 \\ \quad\ -4v_2 = -6v_3 \end{cases} \Rightarrow v = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

$$\Rightarrow E_2(\lambda=-1) = \mathcal{L}\left\{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\right\}$$

Es gilt

$$\lambda_1 = \lambda_2 = 1 \; ; \; v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_3 = -1 \; ; \; v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Es gilt $A = X D X^{-1}$ mit

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

Probe: $AX \stackrel{?}{=} XD$

Andere Möglichkeit

$$D = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

$$AX = \begin{pmatrix} 0 & -1 & 1 \\ -3 & -2 & 3 \\ -2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -2 \end{pmatrix}$$

$$XD = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & -2 \end{pmatrix}$$

= ok ✓