The Meaning of Feynman Diagrams

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Abstract

Feynman diagrams are often used as a 'cartoon' to explain the physics process being studied in a particular reaction. However, they are also a summary of the mathematical calculation that needs to be made in order to predict the rate of that reaction. In this set of three lectures, which were given at the CERN High School Teachers' Programme, I attempt to explain this mathematical basis of Feynman diagrams, without introducing many mathetical details. I am very interested to receive comments and suggestions, both from experts and interested non-experts, on how this explanation can be improved.

Introduction

You will encounter Feynman diagrams in almost any popular discussion of particle physics, and certainly in several of the lecture courses of the High School Teachers' (HST) programme. In most of these contexts, they are used in a rather cartoon-like way, to illustrate the physical process under discussion. However, mention is usually made of the fact that they mean much more than this – they are a summary of the precise mathematical ingredients that must be put together in order to calculate the properties of that process, in particular its rate.

In discussion with the teachers in previous years' HST programmes, it became clear that some of them were uncomfortable with this – they were nervous in using Feynman diagrams as cartoons, because they were hazy on the mathematical foundation that supposedly underlies them. In these lectures I will attempt to bridge this gap, to present the meaning of Feynman diagrams. It is a formidable task, essentially to present the complex mathematical basis of the whole of particle physics, called quantum field theory, without introducing any complex mathematics. It is inevitable therefore that the mathematical level of these lectures is higher than most of the others in the HST programme.

I have tried to organize the material in a progressive order of complexity – the first lecture is mostly pictures and ideas, and at the end I will review some of the mathematics that I will need in the second lecture. That will introduce the key equations, and the so-called 'Feynman rules in x space', and will show that the Feynman rules for scattering processes reproduce Coulomb's Law. Finally in the third lecture, I will convert the Feynman rules to 'momentum space', the form in which they are always used in practice, discuss some subtleties related to antiparticles, and a more advanced application of the rules.

In the lectures themselves I merely state the relevant formulae and do not derive them. I am aware that some people might find this unsatisfactory and attempt to give simple proofs of as many results as possible in the appendix. You are then free to study these at your leisure. The approach is mostly that of Halzen and Martin[1], although the explanations are largely my own.

1 Pictures and Ideas

1.1 Feynman Diagrams as Cartoons

$e^+ e^- \rightarrow \mu^+ \mu^-$
$e\mu \rightarrow e\mu$
$n(d) \rightarrow p(u) e^- \bar{\nu}$
Notation: incoming particles on left, outgoing on right.

Lines with arrows for fermions, wavy lines for photons and Ws/Zs, springy lines for gluons.

An aside for experts: I am actually going to cheat in this course, just to simplify the algebra, by ignorring the complications of fermion spin. i.e. I will draw the diagrams correctly, with electrons, muons and quarks shown as fermions (spin-$\frac{1}{2}$), but in performing calculations I will treat them as scalar bosons (spin-0).

1.2 Space Time Diagrams

History of a particle represented by a curve in space-time, with space across the page, time up the page (the opposite of the usual convention for Feynman diagrams). Free motion at constant speed = straight line (gradient=$1/v$). Instantaneous acceleration = corner. Smooth acceleration = limit of infinite number of infinitesimal kicks.

1.3 Particles as Plane Waves

Recall de Broglie: particles are waves, with wavelength $\lambda = h/p$.
Einstein: $E = hf$.
  Angular frequency: $\omega = 2\pi f = E/\hbar$.
Wave number: $k = 2\pi/\lambda = p/\hbar$.
  Recall any wave solution can be written as a linear combination of plane waves.
  We will treat incoming particle beams as plane waves

$$\phi(\mathbf{x}, t) = e^{-i(\omega t - \mathbf{p} \cdot \mathbf{x})}. \tag{1}$$

As the particles scatter off each other they produce new outgoing plane waves.
  Recall fundamental principle of quantum mechanics: probability of a particle being in a given position is (proportional to) the modulus squared of wave function.

1.4 Path Integral Formalism

Recall Young's two-slit experiment at low intensity: light travels as individual photons, but the photons 'travel through both slits'. The path integral formalism is a way of calculating the wave function at a particular point by summing all the possible routes by which it could get there.
  In the next lecture we will calculate the probability that an electron is scattered as it passes through a region of space containing a potential $A(\mathbf{x}, t)$. The amplitude for this will come from summing over all routes the electron could take through the potential, including all possible positions of all possible numbers of instantaneous accelerations.

1.5 Natural Units

In particle physics we always use 'natural units' in which $c = 1$, $\hbar = 1$, $\epsilon_0 = 1$ and $\mu_0 = 1$.

1.6 Four Vectors

In order to write all the subsequent quantities in a reasonably simple form, we have to introduce a certain amount of notation, the first piece of which is the 'four-vector'.
  We write all our vector quanties in a form in which the space–time connection is more obvious. That is, we promote usual three-vectors by combining with an additional quantity (of course it has to be the right quantity) to form a four-vector.
  We denote an index over the four components by a Greek index, eg μ. To point out that the extra component is slightly different from the other three, μ runs from 0 to 3.

Simplest example: space-time 'point': $x^\mu = (ct; \mathbf{x})$, i.e. x^μ is a vector whose 'zero' component is (c times) time, 'one' component is x, 'two' component is y, 'three' component is z. Recall that we use units in which $c = 1$, so I drop c from here on: $x^\mu = (t; \mathbf{x})$.

Most common example we use: four-momentum: $p^\mu = (E; \mathbf{p})$.

Other examples that will be important for us are the four-current-density j^μ and four-potential A^μ. $j^\mu = (\rho; \mathbf{j})$, where ρ is the charge density and $\mathbf{j}$ is the usual current density. $A^\mu = (\phi; \mathbf{A})$, where ϕ is the electric potential and $\mathbf{A}$ is the magnetic vector potential. Note that the electric and magnetic effects have become intimately linked, which will bring us to Maxwell's equations in a minute.

The final example we will use is the four-derivative, written $\partial^\mu \equiv (\frac{\mathrm{d}}{\mathrm{d}t}; -\nabla)$ (notice the minus sign relative to most four-vectors).

The Lorentz transformations tell us how quantities change when we measure them in reference frames that are moving relative to each other. They tell us that four-vectors are different in different frames. However, they also tell us that if we combine two four-vectors in the right way, the combination is the same in all frames. We define this combination to be the dot-product:

$$a \cdot b \equiv a^\mu b_\mu \equiv a^0 b^0 - \mathbf{a} \cdot \mathbf{b}. \tag{2}$$

It is a little inconvenient to write out the minus signs in this formula explicitly so we sometimes encode them into a matrix:

$$a^\mu b_\mu \equiv a^\mu g_{\mu\nu} b^\nu; \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3}$$

Note that we also use "Einstein's summation convention" in which any index that appears twice is automatically summed over. Thus for example $a^\mu g_{\mu\nu} b^\nu$ is shorthand for $\sum_{\mu=0}^{3} \sum_{\nu=0}^{3} a^\mu g_{\mu\nu} b^\nu$. By the way, whether an index is up or down does actually matter to the experts, but not here: just to satisfy the experts we should always make sure that any index pair that is summed over has one up and one down, but we don't care which.

Like for three-vectors, we define a^2 as shorthand for $a \cdot a$. A particularly important example is the square of the four-momentum:

$$p^2 = E^2 - \mathbf{p}^2 = m^2. \tag{4}$$

Finally, we can put together several of our four-vectors to write Maxwell's equations in four-vector form:

$$\partial^2 A^\mu = j^\mu. \tag{5}$$

You can easily check that this one equation summarizes the four equations you might be more familiar with. We will use Maxwell's equations in this form in the next lecture.

1.7 Fourier Transforms

In deriving the Feynman rules, we will make extensive use of Fourier transforms (FTs). You may be familiar with the FTs between displacement and frequency in acoustics for example. If a signal as a function of time is given by $f(t)$, its frequency decomposition is

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}t \, f(t) \, \mathrm{e}^{i\omega t}, \tag{6}$$

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where the factor in front is a matter of convention. I prefer this one, because it is symmetrical with the inverse transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, f(\omega) \, e^{-i\omega t}. \tag{7}$$

Since we describe space-time points by four-vectors x^μ, we FT with respect to each of the four components. Since energy and momentum are equal to angular frequency and wave number respectively in natural units, the Fourier companion of x^μ is the four-momentum p^μ,

$$f(p) = \frac{1}{(2\pi)^2} \int d^4x \, f(x) \, e^{ip\cdot x}, \tag{8}$$

$$f(x) = \frac{1}{(2\pi)^2} \int d^4p \, f(p) \, e^{-ip\cdot x}. \tag{9}$$

We will see that the Feynman rules can be expressed either "in x space" or "in momentum space" and the two are simply related by the four-dimensional FT.

2 Equations and Applications

In this lecture, we will set up the basic formalism of Feynman diagrams, use it to calculate the probability for one particle to scatter another, note that it can be decomposed into a set of standard components, the Feynman rules, and check that this description reproduces the familiar Coulomb Law.

2.1 Perturbation Theory

We are going to start by calculating the probability that a charged particle is scattered as it travels through a region of space in which there is an electromagnetic potential $A^\mu(x)$. Recall that we said last time that we would view this process as a sum of all possible numbers of pointlike scatters, integrated over all possible positions for those scatters. We now introduce one further idea: that scattering is rare (we will see that it is parametrized by the electric charge of the particle, q, so we assume that q is small). Thus, the most likely outcome is that the particle flies straight through this region of space without scattering. Occasionally it undergoes one scattering, even more occasionally two, and so on. Therefore to calculate the total probability of scattering onto a particular path, a reasonable first approximation is the probability that it does it by one point-like scatter.

2.2 Electron Scattering Off a Fixed Potential

In only three lectures, I am going to have to pluck a formula out of thin air at some point without deriving it. In order to motivate it a little, I first discuss the definition of the current density in terms of the plane wave solution for a freely propagating particle of fixed momentum. One can show that

$$j_\mu(x) \equiv i\,q\Big(\phi^*(x)(\partial_\mu\phi(x)) - (\partial_\mu\phi^*(x))\phi(x)\Big), \tag{10}$$

satisfies the right properties to be the four current density: $\partial_\mu j^\mu = 0$, or, since $j^\mu = (\rho, \mathbf{j})$,

$$\frac{d\rho}{dt} = -\nabla \cdot \mathbf{j}. \tag{11}$$

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That is, the rate of change of charge density at a point is minus the total charge flowing away from that point. In Eq. (10), q is the charge of the particle, $q = -e$ for the electron.

Putting the wave function of a plane wave, given in the last lecture, into Eq. (10), we obtain

$$j_\mu = 2q\,p_\mu = 2E\,q\,(1; \mathbf{v}). \tag{12}$$

Apart from the unusual normalization, (this wave function corresponds to $2E$ particles per unit volume), this has the form expected of a current density: the current density per particle is given by the charge per particle times their velocity.

Now I am going to use Eq. (10) to motivate the form of what I will call the 'transition current', describing the transition from a wave function with initial momentum p_i to another with final momentum p_f,

$$j_\mu^{fi}(x) \equiv i\,q\left(\phi_f^*(x)(\partial^\mu \phi_i(x)) - (\partial^\mu \phi_f^*(x))\phi_i(x)\right) \tag{13}$$

$$= q(p_i + p_f)_\mu\,e^{i(p_f - p_i)\cdot x}. \tag{14}$$

Finally, I pull my formula out of thin air. I claim that the transition amplitude for scattering an initial electron i to a final electron f is given by

$$T_{fi} = -i\int \mathrm{d}^4x\, j_\mu^{fi}(x)\,A^\mu(x). \tag{15}$$

The initial factor of $-i$ is just a matter of convention that I will not justify. The integral over x is motivated by the path integral formalism: we must integrate over all possible positions that the electron can be instantaneously accelerated. The integrand is the potential energy density of the interaction between the electron and the field: the 'zero' components are charge density times electric potential, while the other components are electric current density times magnetic potential. With a little work, it is possible to show that a potential term $q\,p \cdot A$ leads directly to a force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{16}$$

i.e. the Lorentz force.

Even if you didn't get the formulae, it is useful to have the following picture in mind:
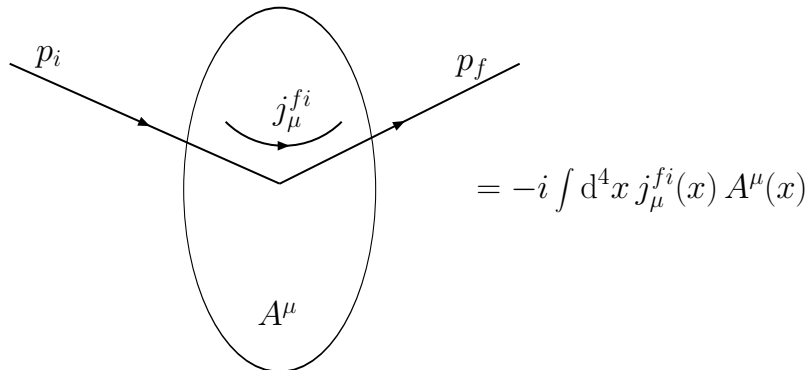
This is our first Feynman diagram and the Feynman rules for it are:

1. Label the space-time position of the scattering as x and integrate over all values of x.

2. When a particle with momentum p_i is scattered to momentum p_f at point x, include a factor of the transition current $j_\mu^{fi}(x)$.

3. When the scattering is due to a potential $A^\mu(x)$, include a factor of $A^\mu(x)$.

2.3   Electron–Muon Scattering

We now consider a scattering process in which one particle, A, scatters another, B, for example electron–muon scattering. B is an accelerated electric charge, so it generates an electromagnetic potential. Let us call this $A_B^\mu(x)$. If we knew this, we could calculate the scattering amplitude exactly as in the previous section:

$$T_{fi} = -i \int \mathrm{d}^4x \, j_\mu^A(x) \, A_B^\mu(x), \tag{17}$$

but what is $A_B^\mu(x)$? It is due to particle B, integrated over all possible positions of B. Information is carried from there to x by the photon field, so let us study how photons propagate. The Maxwell equation we gave in the last lecture is true in general, so is true of our potential in particular:

$$\partial^2 A_B^\mu = j_B^\mu, \tag{18}$$

where j_B^μ is the transition current for particle B. We can solve this equation by Fourier transforming it:

$$-q^2 A_B^\mu(q) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4y \, j_B^\mu(y) \, \mathrm{e}^{iq\cdot y} \tag{19}$$

$$\Rightarrow A_B^\mu(q) = \frac{-1}{q^2} \frac{1}{(2\pi)^2} \int \mathrm{d}^4y \, j_B^\mu(y) \, \mathrm{e}^{iq\cdot y} \tag{20}$$

$$\Rightarrow A_B^\mu(x) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \, \mathrm{e}^{-iq\cdot x} \frac{-1}{q^2} \int \mathrm{d}^4y \, j_B^\mu(y) \, \mathrm{e}^{iq\cdot y}. \tag{21}$$

Finally, rewriting the order of the integration, we have

$$A_B^\mu(x) = \int \mathrm{d}^4y \, j_B^\mu(y) \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{-1}{q^2} \, \mathrm{e}^{iq\cdot(y-x)}. \tag{22}$$

That is, the potential at point x is indeed the integral over all positions particle B could be at, y, times a factor that 'propagates' that information from y to x via a photon field,

$$D^{\mu\nu}(x, y) = \int \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{-g^{\mu\nu}}{q^2} \, \mathrm{e}^{iq\cdot(y-x)}. \tag{23}$$

The scattering amplitude is therefore

$$T_{fi} = -i \int \mathrm{d}^4x \, \mathrm{d}^4y \, j_\mu^A(x) \, D^{\mu\nu}(x, y) \, j_\nu^B(y). \tag{24}$$

It is a useful cross-check of this formula that it is symmetrical in A and B. That is, we could have reversed the argument and calculated the amplitude for B to scatter in the potential of A. In checking this property, it is useful to note that $D^{\mu\nu}(x, y)$ is symmetric in $x \leftrightarrow y$ because the integral in its definition is unchanged by $q \to -q$.

We are now ready to draw this as a Feynman diagram,

$$= -i \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, j_\mu^A(x) \, D^{\mu\nu}(x,y) \, j_\nu^B(y)$$
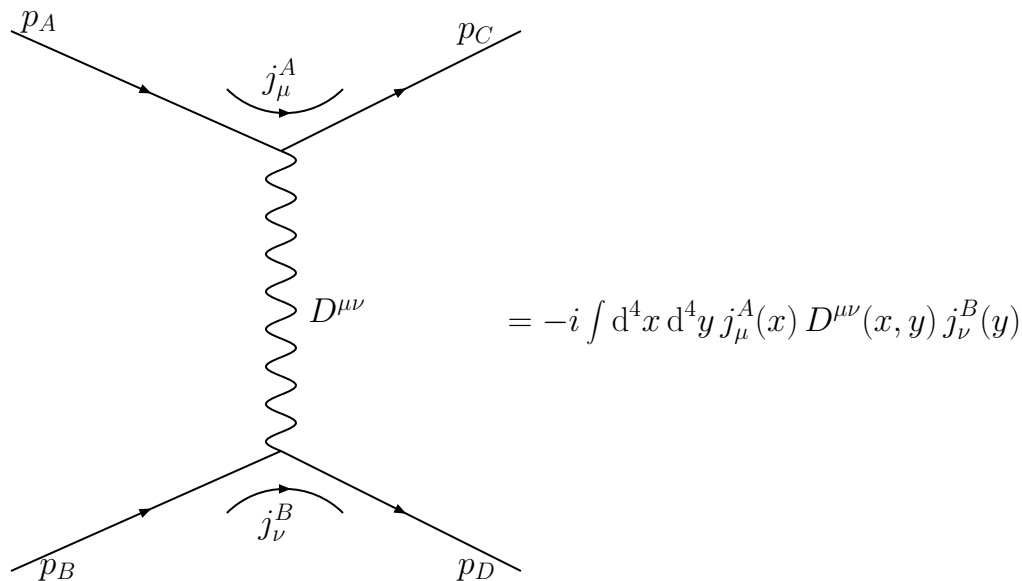
and write down the Feynman rules for arbitrary particle* scattering processes.

2.4 Feynman Rules in 'x' Space

1. Draw all[†] distinct Feynman diagrams.

2. Label the space time points of all vertices and integrate over them.

3. When a particle with momentum p_i is scattered to momentum p_f at point x, include a factor of the transition current $j_\mu^{fi}(x)$.

4. When a photon propagates from some point x to some other point y, include a factor of $D_{\mu\nu}(x,y)$.

We have not yet learnt what to do with diagrams with internal electron lines, i.e. electron propagators, which we will rectify in the final lecture, but otherwise these rules are complete and give us the transition amplitude for *any* scattering process.

2.5 Coulomb's Law

As a check of our understanding of the rules, we can use them to calculate the scattering of an electron by a heavy particle, for example a nucleus, from which we should see Coulomb's Law emerge.

The scattering amplitude is

$$T_{fi} = -i \int \mathrm{d}^4 x \, \mathrm{d}^4 y \, j_\mu^A(x) \, D^{\mu\nu}(x,y) \, j_\nu^B(y). \tag{25}$$

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*Note that we haven't yet discussed antiparticles, or external photons – we will do both in the final lecture.
[†]With certain restrictions that are more (all parts of the diagram must be connected) or less easy to explain.

Expanding out the definition of $D^{\mu\nu}(x,y)$, we have

$$T_{fi} = -i \int \mathrm{d}^4x \, \mathrm{d}^4y \, \frac{\mathrm{d}^4q}{(2\pi)^4} \frac{-j^A \cdot j^B}{q^2} \, \mathrm{e}^{iq\cdot(y-x)}. \tag{26}$$

This expression is Lorentz invariant (i.e. it has the same form in any frame of reference). However, to get insight into the physics it contains, we can choose to work in any particular reference frame we like. In the 'lab' frame in which the heavy particle is initially at rest, it is possible to show that

$$\frac{-j^A \cdot j^B}{q^2} = \frac{j_0^A j_0^B}{|\mathbf{q}|^2}. \tag{27}$$

Recall that the zero component of the four-current is the charge density.

Since the integrand is now independent of q_0, the q_0 integral can be performed: it gives a delta-function forcing x and y to have the same time component. We finally use the fact that $1/|\mathbf{q}|^2$ is the Fourier transform of $1/4\pi|\mathbf{x}|$:

$$\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} \, \mathrm{e}^{i\mathbf{q}\cdot(\mathbf{y}-\mathbf{x})} = \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, \tag{28}$$

to give

$$T_{fi} = -i \int \mathrm{d}^4x \, \mathrm{d}^3\mathbf{y} \, \frac{j_0^A(t,\mathbf{x})j_0^B(t,\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{29}$$

If $j_0^A(t,\mathbf{x})$ represents a particle of charge q_A at a point $\mathbf{x}(t)$, then

$$\int \mathrm{d}^3\mathbf{x} \, j_0^A(t,\mathbf{x}) = q_A \tag{30}$$

and we have

$$T_{fi} = -i \int \mathrm{d}t \, \frac{q_A q_B}{4\pi|\mathbf{x}(t)-\mathbf{y}(t)|}. \tag{31}$$

That is the scattering takes place through the Coulomb's Law potential. Note the sign: the potential is positive and decreasing with distance if the charges have the same sign, corresponding to a repulsive force and vice versa if the charges are opposite.

2.6 Summary

We have found that the transition amplitude for a given scattering process can be obtained by following the Feynman rules. In particular, we must draw all possible scattering diagrams and attach given factors corresponding to each part (internal and external lines and vertices). The factor for an internal photon line, called the photon propagator, describes the propagation of a 'virtual' photon of momentum q from one vertex of the diagram to another. Unlike a 'real' photon it does not have $q^2 = 0$, so cannot travel over macroscopic distances. In the final lecture we will try to understand more about what this means, and construct the equivalent propagator for an electron, before studying a couple of more advanced applications of the Feynman rules.

3 Antiparticles and Advanced Applications

In the final lecture we clear up a couple of final points related to the photon propagator and the Feynman rules, before discussing antiparticle wave functions and particle propagators in detail. We look at an application of the particle propagator in calculating the scattering of photons by electrons. Finally we look at perturbation theory beyond leading order and Feynman diagrams containing closed loops.

3.1 Physical Interpretation of the Photon Propagator

Before moving on, we discuss the photon propagator $D^{\mu\nu}(x,y)$ in more detail, to try to get a better understanding of what it represents.

First I point out, as I mentioned in the last lecture, the fact that it is symmetric in x and y, which will be important shortly.

Second, we can shed light on the physical role of the propagator, by performing the integral over the energy component of q. To make the result more general and anticipate the particle propagator we will derive later, we include a mass m. The photon case can of course be obtained by setting $m = 0$. The energy integral turns out to give different results, depending on whether the time component of x (t_x) is before or after that of y (t_y):

$$D(x,y) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2} \, \mathrm{e}^{iq\cdot(y-x)} = \int \frac{\mathrm{d}^3 \mathbf{q}}{2E_q \, (2\pi)^3} \begin{cases} \mathrm{e}^{-iq\cdot(x-y)} & t_x > t_y \\ \mathrm{e}^{-iq\cdot(y-x)} & t_y > t_x \end{cases}, \tag{32}$$

where $E_q = \sqrt{|\mathbf{q}|^2 + m^2}$ is the energy of a particle of mass m and momentum $\mathbf{q}$. Notice that the sign in the exponent is different in the two cases, which gives us a very nice interpretation: if the point x is at a later time than the point y, the propagator describes a photon propagating towards x from y, i.e. forwards in time; if x is at an earlier time than y, it describes a photon propagating towards y from x, i.e. again forwards in time.

Thus the Feynman diagram as written in its usual form is a summary of all possible space-time diagrams, including both possible time-orderings. If we use the definition of q suggested by Eq. (32) we can interpret the photon as always travelling forwards in time:

whereas if we instead always label q as flowing in the same direction, then one of the time ordered

diagrams corresponds to a photon travelling backwards in time:

It is important to note that the time order of the two vertices is frame dependent: if $t_x > t_y$ in one frame then one can view the same scattering event in another in which $t_y > t_x$. Thus we conclude that the only consistent, frame-dependent, way of labeling q is always in the same direction, and we are lead inevitably to the idea of the photon propagating backwards in time in some frames.

3.2   Feynman Rules in Momentum Space

In the previous lecture we formulated the Feynman rules in x space. They involved wave functions for external particles (and hidden inside the photon propagator) as well as integrals over all possible space-time positions of all vertices. It turns out that these integrals always boil down to Fourier transforms, and the same rules can be expressed more simply in momentum space. As an example, we return to electron–muon scattering. Recall that the transition amplitude was

$$T_{fi} = -i \int d^4x \, d^4y \, j_\mu^A(x) \, D^{\mu\nu}(x, y) \, j_\nu^B(y). \tag{33}$$

Expanding everything out and regrouping it, we have

$$T_{fi} = -i \int d^4x \, d^4y \, \frac{d^4q}{(2\pi)^4} \, q_A(p_A + p_C)_\mu \, \frac{-g^{\mu\nu}}{q^2} \, q_B(p_B + p_D)_\nu \, e^{i(p_C - p_A) \cdot x + iq \cdot (y - x) + i(p_D - p_B) \cdot y}. \tag{34}$$

The only x dependence is through the exponential factor, which can be integrated to produce a δ-function enforcing energy-momentum conservation at the vertex, i.e. $q = p_C - p_A$,

$$\int d^4x \, e^{i(p_C - p_A - q) \cdot x} = (2\pi)^4 \delta^4(p_C - p_A - q), \tag{35}$$

and similarly for y, giving $q = -(p_D - p_B)$,

$$\int d^4y \, e^{i(p_D - p_B + q) \cdot y} = (2\pi)^4 \delta^4(p_D - p_B + q). \tag{36}$$

The two δ-functions can then be combined to show that overall energy-momentum is conserved,

$$(2\pi)^4 \delta^4(p_C - p_A - q) \, (2\pi)^4 \delta^4(p_D - p_B + q) = (2\pi)^4 \delta^4(p_C - p_A - q) \, (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \tag{37}$$

and one of them can be used on the q integration,

$$\int \frac{d^4q}{(2\pi)^4} \, (2\pi)^4 \delta^4(p_C - p_A - q) \, f(q) = f(p_C - p_A). \tag{38}$$

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Finally, we have an expression in closed form, with no more integrals to perform,

$$T_{fi} = -i\, q_A(p_A + p_C)_\mu \left. \frac{-g^{\mu\nu}}{q^2} \right|_{q=p_C - p_A} q_B(p_B + p_D)_\nu\, (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D) \qquad (39)$$

and we have derived the fact that overall energy-momentum is conserved (i.e. the transition amplitude is zero for transitions in which it is not conserved). This expression can be directly constructed from the Feynman rules in momentum space:

1. Draw all distinct Feynman diagrams.

2. Label the momenta of all internal lines using momentum conservation at each vertex.

3. For each external particle, include a factor of 1.

4. For each vertex at which a particle with charge q and momentum p_i is scattered to momentum p_f, include a factor of $q(p_i + p_f)^\mu$.

5. For each photon propagator include a factor of $-g_{\mu\nu}/q^2$.

Notice that by transforming to momentum space we have simplified the process, since there are no more integrals to do, but we have also lost information: in the x space approach we derive energy-momentum conservation, while in momentum space we have to impose it by hand. Notice also that in momentum space the factor for an external particle is trivial. This is only true because we are pretending that it is a scalar – external fermions and vector bosons do contribute non-trivial factors.

3.3   Antiparticles

The relativisitic generalization of the Schrödinger equation, called the Klein–Gordon equation,

$$(\partial^2 + m^2)\phi = 0, \qquad (40)$$

discussed in more detail in the appendix, has plane wave solutions

$$\phi = e^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \qquad (41)$$

with

$$E = \pm\sqrt{|\mathbf{p}|^2 + m^2}. \qquad (42)$$

The positive energy solutions are the ones we have already been using in calculating transition amplitudes. Historically, the puzzle of the negative energy solutions is what drove Dirac to consider his equation instead. To his surprise, it turned out to be an equation of spin-$\frac{1}{2}$ particles, and still had negative energy solutions. As we have discussed in some of the other lecture courses, he was able to solve the negative energy problem using the Pauli exclusion principle by invoking the idea of a full sea of negative energy states. He then interpreted holes in this sea, with the lack of a negatively charged negative energy electron, as a positively charged positive energy antielectron.

This was all very well, and won him a Nobel prize, but the fact remains that there are spin-0 particles, for example the pion, and they are governed by the Klein–Gordon equation, but the sea

interpretation is not valid since there is no Pauli exclusion for bosons: the negative energy states can simply decay to even more negative energy.

A more general explanation of negative energy solutions, which works equally well for fermions and bosons, is provided by the Feynman–Stückelberg interpretation. This simply states that a negative energy solution describes a particle that is propagating backwards in time, and that it can be reinterpreted as a positive energy antiparticle propagating forwards in time. We can motivate this by noting that the wave function for a negative energy particle travelling backwards in time contains a factor $e^{-i(-E)(-t)}$ and making the almost trivial observation that this can be rewritten

$$e^{-i(-E)(-t)} = e^{-iEt}. \tag{43}$$

This explains the energy part. To see that the positive energy interpretation of this negative energy state has opposite charge, consider the current density for an electron discussed earlier,

$$j_\mu(e^-) = 2q\, p_\mu = 2q(E; \mathbf{p}), \tag{44}$$

where $q = -e$ is the charge of the electron. If the electron has negative energy then its current density is (continuing to define E to be positive)

$$j_\mu(\text{–ve energy } e^-) = 2q\, p_\mu = 2q(-E; \mathbf{p}) \tag{45}$$
$$\Rightarrow j_\mu(e^+) = -2q(E; -\mathbf{p}) = 2(-q)(-p_\mu). \tag{46}$$

Note that q is still the charge of the electron, $-e$, and the charge of the positive energy positron is $-q = +e$.

The implementation of the Feynman–Stückelberg interpretation is therefore straightforward: we simply have to take the arrow on particles and antiparticles in Feynman diagrams seriously and treat a positron with momentum p as an electron with momentum $-p$. We therefore slightly reword one of the Feynman rules:

4. For each vertex at which a particle with charge q enters the vertex with momentum p_i and leaves with momentum p_f, include a factor of $q(p_i + p_f)^\mu$.

Note that it is always the particle charge q that appears in this expression, even for antiparticle scattering.

As an example, we can now calculate the transition amplitude for $e^+ e^- \to \mu^+ \mu^-$:

Note that the electron and positron momenta are p_A and p_B, so that in the Feynman rule we have $p_i = p_A$ and $p_f = -p_B$, so that the expression for the vertex is $q_e(p_A - p_B)^\mu$.

3.4 The Electron Propagator

Puting together the physical interpretation of the photon propagator and our new understanding of antiparticles, it is rather straightforward to write down and interpret the propagator of a matter particle. Recall that I am cheating by pretending that our particles are actually spin 0, so our particle propagator has no indices. Apart from this and the fact that it has a mass, it is identical to the photon propagator,

$$D(x, y) \; = \; \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{q^2 - m^2} \, e^{iq\cdot(y-x)} \qquad (x \text{ space}), \tag{47}$$

$$D(q) \; = \; \frac{1}{q^2 - m^2} \qquad (p \text{ space}). \tag{48}$$

We can therefore write down one of the Feynman diagrams[‡] for electron–positron annihilation to photons, $e^+ e^- \to \gamma\gamma$,

Note that in the conventional notation for the Feynman diagram we cannot say whether the vertical line is an electron or a positron, because in one time ordering it is an electron, while in the other it is an electron travelling backwards in time, i.e. a positron. Observers in different frames will see different time orderings and therefore different identities for the internal line in the same event.

3.5 Compton Scattering

Compton scattering is $e\gamma \to e\gamma$. The wave functions for the external photons are given by solutions of Maxwell's equations in vacuum and are

$$A^\mu = \epsilon^\mu \, e^{-ip\cdot x}, \tag{49}$$

where ϵ^μ is the polarization vector of the photon. This can be chosen to satisfy $\epsilon \cdot p = 0$ and to have zero energy component in some particular frame of reference. Therefore the photon is purely transverse in this frame.

  We therefore have to add a Feynman rule to our list:

6. For each incoming photon of momentum p include a factor of $\epsilon^\mu(p)$ and for each outgoing photon of momentum p include a factor of $\epsilon^{*\mu}(p)$.

[‡]There are two: you might like to think about what the other one must be. Hint: the two photons in the final state are indistinguishable.

We can therefore draw the diagrams for Compton scattering,

and evaluate the transition amplitude,

$$T_{fi} = -i\, q^2\, \epsilon^{\mu}(p_A)\, \epsilon^{*\nu}(p_D) \left((2p_B + p_A)_{\mu}\, \frac{1}{(p_A + p_B)^2}\, (2p_C + p_D)_{\nu} + (2p_C - p_A)_{\mu}\, \frac{1}{(p_A - p_C)^2}\, (2p_C - p_D)_{\nu} \right).$$

$$(50)$$

Recalling that $\epsilon(p) \cdot p = 0$, this expression can be simplified somewhat.

3.6   Loops

Higher orders of perturbation theory correspond to diagrams with internal loops.
In p space another rule must be added:

7. For each closed loop containing a particle of momentum k include a factor of $\int \frac{\mathrm{d}^4 k}{(2\pi)^4}$.

In x space no new rules are needed.
Loops give rise to real physical effects, for example Lamb shift, $g-2$ of the electron, etc.
Also give rise to the complications of renormalization, cf 't Hooft and Veltman's Nobel Prize.

Summary

In these lectures I have tried to give you a flavour of the mathematics that underlies Feynman diagrams. More importantly, also the physics that underlies them and, in particular propagators. If you got nothing else out of the lectures, I hope you will at least remember the picture of a Feynman diagram as the sum over all time ordered space-time diagrams and of a propagator as the sum of an exchanged particle going backwards or forwards in time.

A   More Mathematical Details

In this appendix, I give some more details of the mathematical steps that I skipped in the lectures, for the benefit of the (probably not many) people in the audience who want them. Some chunks of this are line-for-line identical to the derivations in Halzen and Martin, and I am grateful for their permission to use their text in this way.

A.1 Relativistic Schrödinger Equation and the Current Density

By deriving the relativistic generalization of Schrödinger's Equation, called the Klein–Gordon equation, we can derive the expression for the current density, Eq. (10).

A.2 Motivation for the Transition Amplitude

By studying the Schrödinger equation it is possible to motivate the form of the transition current, Eq. (13) and transition amplitude, Eq. (15).

A.3 From $\phi^* V \phi$ To $j \cdot A$

For people who know Gauss' theorem, this is only a few steps.

A.4 From $\rho\, v \cdot A$ To $q\, v \times B$

This is a slightly more advanced step, which needs Lagrangian mechanics.

A.5 From $-j^A \cdot j^B / q^2$ To $j_0^A j_0^B / |\mathrm{q}|^2$

This exploits the 'gauge invariance' of j, which results in $j \cdot q = 0$.

References

[1] F. Halzen and A.D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics*, John Wiley, New York (1984).