CHAPTER II: The QCD Lagrangian

2.1. Preparation: Gauge invariance for QED

- Consider electrons represented by Dirac field $\psi(x)$. Gauge transformation:

$$\psi(x) \to U\psi(x) \text{ with } U = e^{-i\theta} \tag{2.1}$$

  – Local gauge transformation, if $\theta = \theta(x)$

  – Global gauge transformation, if $\theta = const.$

Hypothesis : Local gauge transformations, $U = e^{-i\theta(x)}$, leave the physics invariant.

- Current is invariant under local gauge transformation.

$$\bar{\psi}(x)\gamma_\mu\psi(x) \xrightarrow{G.T.} \psi^\dagger\gamma_0 U^\dagger\gamma_\mu U\psi \tag{2.2}$$

- Not invariant:

$$\begin{aligned} \bar{\psi}i\gamma_\mu\partial^\mu\psi \to \bar{\psi}i\gamma_\mu U^\dagger\partial^\mu(U\psi) \\ = \bar{\psi}i\gamma_\mu U^\dagger U(\partial^\mu\psi) + \bar{\psi}i\gamma_\mu\psi\underbrace{(U^\dagger i\partial^\mu U)}_{\partial^\mu\theta(x)} \end{aligned} \tag{2.3}$$

- Introduction of gauge field $A^\mu(x)$:

  Definition of gauge covariant derivative: $D^\mu = \partial^\mu - ieA^\mu(x) \quad (e > 0)$

- Requirement: Under local gauge transformation

$$\widetilde{D^\mu\psi} = U(D^\mu\psi)$$

  then $\mathcal{L}' = \bar{\psi}\big(i\gamma_\mu D^\mu - m\big)\psi$ gauge invariant.

$$\begin{aligned} U\big(D^\mu\psi\big) &= \partial^\mu\tilde{\psi} - ie\tilde{A}^\mu(x)\tilde{\psi} = \partial^\mu\big(U\psi(x)\big) - ie\tilde{A}^\mu(x)U\psi(x) \\ &= \big(\partial^\mu U\big)\psi + U\big(\partial^\mu\psi\big) - ie\tilde{A}^\mu(x)U\psi \\ &= U\big[\partial^\mu - ieA^\mu(x)\big]\psi(x) \end{aligned} \tag{2.4}$$

$$\Rightarrow -ie\tilde{A}^\mu U\psi = -ieUA^\mu\psi - \big(\partial^\mu U\big)\psi$$

$$\Rightarrow \tilde{A}^\mu U = UA^\mu - \frac{i}{e}\partial^\mu U$$

$$\begin{aligned} \tilde{A}^\mu &= UA^\mu U^\dagger - \frac{i}{e}(\partial^\mu U)U^\dagger \\ &= U\Big[A^\mu - \frac{i}{e}U^\dagger\partial^\mu U\Big]U^\dagger \end{aligned} \tag{2.5}$$

- Gauge field $\leftrightarrow$ Potentials: $A^\mu(x) = \big(\phi(x),\, \vec{A}(x)\big)^T$.

- Electromagnetic fields:

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla}\times\vec{A} \end{aligned}$$

- Electromagnetic field tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.6}$$

- Lagrangian density of electromagnetic fields

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) = -\frac{1}{2}\big(\vec{E}^{\,2} - \vec{B}^{\,2}\big) \tag{2.7}$$

- Equations of motions for free photon: $\Box A^\mu(x) = 0$

$$A^\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3k}{(2\pi)^3\, 2\omega_k}\Big[a(k,\lambda)\,\epsilon^\mu_{(\lambda)}\,e^{-ik\cdot x} + a^\dagger(k,\lambda)\,\epsilon^{\mu^*}_{(\lambda)}\,e^{ik\cdot x}\Big] \tag{2.8}$$

  where $\omega_k = |\vec{k}|$ and $\epsilon^\mu_{(\lambda)}$ represents the polarization vector.

- State vector of photon:

$$\begin{aligned} |k,\,\lambda\,\rangle &= a^\dagger(k,\,\lambda)|0\rangle \\ a(k,\,\lambda)|k,\,\lambda\,\rangle &= |0\rangle \end{aligned} \tag{2.9}$$

- Lagrangian density of QED:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(x)\Big[i\gamma_\mu D^\mu - m\Big]\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \tag{2.10}$$

  where $D^\mu = \partial^\mu - ieA^\mu(x)$

- Gauge transformations form a group: $U = e^{-i\theta(x)}$ (QED), $U \in Group\ U(1)$.

2.2. Local $SU(3)$ Gauge transformations

- Starting point: Quark fields $\psi = (\psi_{\alpha i})$

$$\begin{cases} \alpha = u,\, d,\, s,\, c,\, b,\, t \;\; \text{(flavor index)} \;\; N_f = 6 \;\longleftrightarrow\; SU(N_f) \\ i = 1,\, 2,\, 3 \qquad\qquad\quad \text{(color index)} \;\; N_c = 3 \;\longleftrightarrow\; SU(3)_c \end{cases}$$

where $\psi_{\alpha i}$ is a 4-component Dirac-spinor.

Consider Quark fields with color degree of freedom and their free *Lagrangian*:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \mathcal{L}_0 = \bar{\psi}\left[\, i\gamma_\mu \partial^\mu - m\right]\psi \qquad (2.11)$$

- Local $SU(3)_c$ gauge transformations

$$\psi(x) \;\longrightarrow\; \tilde{\psi}(x) = U\,\psi(x) \qquad (2.12)$$

with $U = \exp\left[-\,i\,\theta_a(x)\,\frac{\lambda_a}{2}\right]$ where $\theta_a(x)$ is a real function with $a = 1,\, 2,\, \cdots,\, 8$.

Hypothesis : Physics of strong interaction of quarks is invariant under gauge transformation: $\psi(x) \to U(x)\,\psi(x)$.

$SU(3)_c$ is a *non-abelian* gauge group.

- Gauge covariant derivative:

$$D_\mu = \partial_\mu - i\,g\,A_\mu(x) \qquad (2.13)$$

where g is a dimensionless coupling strength analogous to e in QED.

$$A_\mu(x) = \sum_{a=1}^{8} t_a\, A^a_\mu(x) \;\;, \qquad t_a = \frac{\lambda_a}{2} \qquad (2.14)$$

Introducing $A^a_\mu(x)$, $SU(3)_c$ gauge fields "*gluons*",

$$\mathcal{L}_1 = \bar{\psi}(x)\left[i\gamma_\mu D^\mu - m\,\right]\psi(x) \qquad (2.15)$$

Lagrangian $\mathcal{L}_1$ becomes gauge invariant.

$$\boxed{\begin{aligned} \widetilde{D^\mu\psi} &\equiv \partial^\mu\tilde{\psi} - i\,g\,\tilde{A}^\mu\tilde{\psi} = U\left(D^\mu U\right) \\ \tilde{A}^\mu &= U\left[\,A^\mu - \frac{i}{g}\,U^\dagger\,\partial^\mu U\,\right]U^\dagger \end{aligned}} \qquad (2.16)$$

- Infinitesimal gauge transformation

$$U = \exp\left[\,-\,i\,\theta_a(x)\,t_a\,\right] \simeq 1 - i\,\theta_a(x)\,t_a + \cdots \qquad (2.17)$$

transformation of gauge field up to terms linear in $\theta_a(x)$

$$\boxed{A^\mu_a(x) \;\to\; \tilde{A}^\mu_a(x) = A^\mu_a(x) - \frac{1}{g}\,\partial^\mu\theta_a(x) + f_{abc}\theta_b(x)A^\mu_c(x)} \qquad (2.18)$$

- Gluons are massless (a mass term $m_g A^\mu_a A^a_\mu$ would not be gauge invariant).

- Gluonic field tensors:

If one would take the form analogous to QED,

$$F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x), \qquad (2.19)$$

not gauge invariant in QCD.

Introduce additional term to obtain gauge invariant Gluonc field tensor.

$$\boxed{G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g\,f_{abc}\,A^b_\mu(x)\,A^c_\nu(x)} \qquad (2.20)$$

$$G_{\mu\nu} \equiv t_a\,G^a_{\mu\nu} = \frac{i}{g}\left[\,D_\mu\,,\,D_\nu\,\right] \qquad (2.21)$$

- Gluonic Lagrangian:

$$\mathcal{L}_{\rm glue} = -\frac{1}{4}\,G^a_{\mu\nu}(x)\,G^{\mu\nu}_a(x) = -\frac{1}{2}\,{\rm tr}\{G_{\mu\nu}\,G^{\mu\nu}\} \qquad (2.22)$$

2.3. QCD Lagrangian

- QCD Lagrangian:

$$\boxed{\mathcal{L}_{\rm QCD} = \bar{\psi}\,\left(i\gamma_\mu D^\mu - m\right)\psi - \frac{1}{2}\,{\rm tr}\{G_{\mu\nu}\,G^{\mu\nu}\}} \qquad (2.23)$$

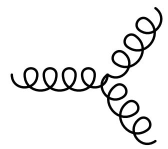
with $D^\mu = \partial^\mu - igA^\mu(x)$.[1]

[1] Remark : frequently $A^\mu \;\to\; gA^\mu$

$$\Rightarrow\; \mathcal{L}_{\rm QCD} = \bar{\psi}\,(i\gamma_\mu(\partial^\mu - iA^\mu) - m)\,\psi - \frac{1}{2g^2}\,{\rm tr}\{G_{\mu\nu}\,G^{\mu\nu}\}$$

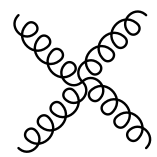
- Gluonic field tensor of $\mathcal{L}_{\mathrm{QCD}}$ generates non-linear gluon interactions:
  - 3-gluon interaction

$$\mathcal{L}^{(3)} = -\frac{g}{2}\, f_{abc}\left(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu\right) A_\mu^b A_\nu^c \qquad \sim g \tag{2.24}$$



  - 4-gluon interaction

$$\mathcal{L}^{(4)} = -\frac{g^2}{4}\, f_{abc}\, f_{cde}\, A_{a\mu} A_{b\nu} A_c^\mu A_d^\nu \qquad \sim g^2 \tag{2.25}$$



2. 4. Classical QCD equation of motion

- Euler-Lagrange equations derived from $\mathcal{L}_{\mathrm{QCD}}(\psi,\, \partial_\mu\psi,\, A_\mu,\, \cdots)$

$$\frac{\partial \mathcal{L}_{\mathrm{QCD}}}{\partial q_i} - \partial_\mu \frac{\partial \mathcal{L}_{\mathrm{QCD}}}{\partial(\partial_\mu q_i)} = 0 \tag{2.26}$$

  - Equations of motion for quark field:

$$\left[i\gamma_\mu\left(\partial^\mu - igA^\mu(x)\right) - m\right]\psi = 0 \tag{2.27}$$

  - Equations of motion for gluon field:

$$\partial^\mu G_\mu^a(x) + g\, f_{abc}\, A_b^\mu(x) G_{\mu\nu}^c(x) = -g\, J_\nu^a(x) \tag{2.28}$$

  with color currents of quarks

$$J_\nu^a(x) = \bar{\psi}(x)\,\gamma_\nu\, t_a\, \psi(x) = \bar{\psi}\,\gamma_\nu \frac{\lambda_a}{2}\psi \tag{2.29}$$

  which are conserved: $\partial^\mu J_\mu^a(x) = 0$.

2. 5. Gauge fixing

- Digression on gauge fixing in electrodynamics:

$$\mathcal{L}_\gamma = -\frac{1}{4}\, F_{\mu\nu} F^{\mu\nu} \tag{2.30}$$

Corresponding equation of motion:

$$\begin{aligned} \partial^\mu F_{\mu\nu}(x) &= \partial^\mu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\ &= \Box A_\nu - \partial_\nu(\partial^\mu A_\mu) \\ &= 0 \end{aligned} \tag{2.31}$$

Gauge theories have a certain freedom in defining the gauge field, $A^\mu(x)$.

In order to remove the problem, eliminate the gauge freedom by setting constraints for the field $A^\mu(x)$.

For example,

$$\boxed{\partial^\mu A_\mu(x) = 0} \tag{2.32}$$

which is called "Lorenz gauge" (covariant constraint).

- Introduce extra term $\lambda\left(\partial_\mu A_a^\mu(x)\right)^2$ with Lagrange multiplier parameter $\lambda = -\frac{1}{2\xi}$

$$\boxed{\mathcal{L}_\gamma = -\frac{1}{4}\, F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{2\xi}(\partial^\mu A_\mu(x))^2} \tag{2.33}$$

Equation of motion

$$\Box A^\mu - \left(1 - \frac{1}{\xi}\right)\partial^\mu(\partial_\lambda A^\lambda) = 0 \tag{2.34}$$

- Gauge fixing choices $\begin{cases} \xi = 1 \ ; \text{Feynman gauge} \\ \xi = 0 \ ; \text{Landau gauge} \end{cases}$

Other options:

$$\begin{aligned} \vec{\nabla}\cdot\vec{A}_a &= 0 \ ; \text{Coulomb gauge} \\ A_a^3 &= 0 \ ; \text{Axial gauge} \\ A_a^0 &= 0 \ ; \text{Temporal gauge} \end{aligned}$$

Appendix: $SU(N)$-Group and Lie algebra

Short mathematical appendix about groups:

- Group: $G = \{g,\, h,\, k,\, \cdots\}$
  - For $g,\, h \in G$, $gh \in G$
  - There exists a "unit" element e such that $eg = ge = g$.
  - For each $g \in G$, there exists an inverse $g^{-1} \in G$; $g^{-1}g = gg^{-1} = e$.
- Linear group:

  Elements $g,\, h,\, \cdots$ (transformations/operators) with the following property:

  For each $g,\, h \in G$ exists $\alpha g + \beta h \in G$ with $\alpha,\, \beta \in \mathbb{C}$
- Representations of a linear group:

  Mapping: $g \in G \to (a_{ij}) \in$ space of complex valued matrices with $a_{ij} \in \mathbb{C}$.
- Adjoint operator:

  Let $g \in G$ (linear), then there exists a unique $g^\dagger$ with the representation $(a_{ij})^\dagger = (a^*_{ji})$.
- Unitary transformations/operators: $U \in G$

$$U^\dagger = U^{-1} \;\Rightarrow\; U^\dagger U = UU^\dagger = \mathbb{1}. \tag{2.35}$$

Consequently a unitary transformation can be written as follows:

$$U = \exp[\,iH\,] = \mathbb{1} + iH + \frac{i^2}{2}H^2 + \cdots \tag{2.36}$$

with Hermitian operator H, *i.e.* $H^\dagger = H$.

Example-1. Group $U(1)$ with elements $U = \exp[i\alpha]$ where $\alpha \in \mathbb{R}$

$$U^\dagger = e^{-i\alpha}\,, \quad UU^\dagger = U^\dagger U = \mathbb{1}$$

Group of gauge transformation in QED

Example-2. Group $SU(N)$

Group of unitary transformations represented by unitary $N \times N$ matrices

$$U = \exp\left[i\sum_a \alpha_a X_a\right] \;\text{ with }\; |\det U|^2 = 1$$

where α_a are real parameters with $a = 1, \cdots, N^2-1$. The hermitian operators X_a are the generators of the $SU(N)$ group.

Generators form Lie-algebra:

$$\boxed{[X_a\,,\,X_b\,] = i\,f_{abc}\,X_c} \tag{2.37}$$

where f_{abc} are the structure constants of the group.

▷ For $N = 2$, $SU(2)$ generators $X_a = \sigma_a/2$ $(a = 1, 2, 3)$

Pauil matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.38}$$

$$\begin{aligned} \mathrm{tr}\{\sigma_a\} &= 0 \\ \mathrm{tr}\{\sigma_a\,\sigma_b\} &= 2\,\delta_{ab} \end{aligned} \tag{2.39}$$

Structure constants: $f_{abc} = \epsilon_{abc}$.

▷ For $N = 3$, $SU(3)$ generators $X_a = \lambda_a/2$ $(a = 1, \cdots, 8)$

Gell-Mann matrices:

$$\begin{aligned} &\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ &\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\; \lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{2.40}$$

$$\mathrm{tr}\{\lambda_a\} = 0$$
$$\mathrm{tr}\{\lambda_a \lambda_b\} = 2\,\delta_{ab} \tag{2.41}$$

Lie-algebra:

$$\left[\lambda_a\,,\,\lambda_b\right] = 2\,i\,f_{abc}\,\lambda_c \tag{2.42}$$

Structure constants:

$$f_{abc} = -i\,\mathrm{tr}\left(\left[\frac{\lambda_a}{2}\,,\,\frac{\lambda_b}{2}\right]\lambda_c\right) \tag{2.43}$$

f_{abc} is totally antisymmetric with nonvanishing members,

$$\begin{aligned} &f_{123} = 1 \\ &f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2} \\ &f_{458} = f_{678} = \sqrt{\frac{3}{2}} \end{aligned} \tag{2.44}$$

- Irreducible representations of $SU(2)$:

$$X_a \equiv J_a = \frac{\sigma_a}{2} \quad (a = 1,\,2,\,3)$$

  – Casimir operator of $SU(2)$: $J^2 = J_1^2 + J_2^2 + J_3^2$

  which commutes with all generators

$$\left[J^2\,,\,J_a\right] = 0 \quad (a = 1,\,2,\,3). \tag{2.45}$$

  – Ladder (raising and lowering) operators:

$$\begin{aligned} J_\pm &= J_1 \pm iJ_2 \\ J^2 &= \frac{1}{2}\left(J_+J_- + J_-J_+\right) + J_3^2 \\ \left[J_+\,,\,J_-\right] &= 2\,J_3\,, \quad \left[J_3\,,\,J_\pm\right] = \pm J_\pm \end{aligned} \tag{2.46}$$

  – Eigenstates of J^2 and J_3 :

$$J^2\,|\lambda,\,M\rangle = \lambda\,|\lambda,\,M\rangle\,, \quad J_3\,|\lambda,\,M\rangle = M\,|\lambda,\,M\rangle \tag{2.47}$$

$$J^2 - J_3^2 = J_1^2 + J_2^2 \geq 0 \implies \lambda - M^2 \geq 0 \tag{2.48}$$

  – Let j be the largest M: $J_+\,|\lambda,\,j\rangle = 0$

$$\begin{aligned} J_-J_+\,|\lambda,\,j\rangle &= \left(J^2 - \frac{1}{2}\left[J_+\,,\,J_-\right] - J_3^2\right)|\lambda,\,j\rangle \\ &= \left(J^2 - J_3 - J_3^2\right)|\lambda,\,j\rangle \\ &= \left(\lambda - j^2 - j\right)|\lambda,\,j\rangle \\ &= 0. \end{aligned} \tag{2.49}$$

  Therefore

$$\lambda = j(j+1) \geq 0. \tag{2.50}$$

  – Relabeling the states $|\lambda,\,M\rangle \equiv |\,j,\,M\rangle$, Eq. (2.47) becomes

$$J^2\,|\,j,\,M\rangle = j(j+1)\,|\,j,\,M\rangle\,, \quad J_3\,|\,j,\,M\rangle = M\,|\,j,\,M\rangle. \tag{2.51}$$

  – Let j' be the smallest M: $J_-\,|\,j,\,j'\rangle = 0$

$$\begin{aligned} J_+J_-\,|\,j,\,j'\rangle &= \left(J^2 + J_3 - J_3^2\right)|\,j,\,j'\rangle \\ &= \left(j^2 + j + j' - j'^2\right)|\,j,\,j'\rangle \\ &= 0. \end{aligned} \tag{2.52}$$

  Hence

$$j(j+1) = j'(j'-1) \implies j' = -j. \tag{2.53}$$

  – Basis states:

  $\{\;|\,j,\,M\rangle$ with $M = j,\,j-1,\,\cdots,\,-j$, dimension: $d_j = 2j+1\;\}$.

- Product of representations of $SU(2)$:

$$J = J^{(1)} + J^{(2)}\,, \quad J_3 = J_3^{(1)} + J_3^{(2)} \tag{2.54}$$

$$\begin{aligned} {J^{(i)}}^2\,|\,j^{(i)},\,M^{(i)}\rangle &= j^{(i)}(j^{(i)}+1)\,|\,j^{(i)},\,M^{(i)}\rangle \\ J_3^{(i)}\,|\,j^{(i)},\,M^{(i)}\rangle &= M^{(i)}\,|\,j^{(i)},\,M^{(i)}\rangle. \end{aligned} \tag{2.55}$$

To look for $|\,j,\,M\rangle$ with $J^2\,|\,j,\,M\rangle = j(j+1)\,|\,j,\,M\rangle$ and $J_3\,|\,j,\,M\rangle = M\,|\,j,\,M\rangle$, in general, we form appropriate linear combinations of product states:

$$|\,j,\,M\rangle = \sum_{M^{(1)},\,M^{(2)}} \left(j^{(1)}M^{(1)}j^{(2)}M^{(2)}|\,jM\right)\,|\,j^{(1)},\,M^{(1)}\rangle\,|\,j^{(2)},\,M^{(2)}\rangle \tag{2.56}$$

where the quantities $\left(j^{(1)}M^{(1)}j^{(2)}M^{(2)}|\,jM\right)$ are called Clebsch-Gordan coefficients.

Example. Coupling of two states in "fundamental" representation of $SU(2)$; basis states $\{\, |\, j^{(i)} = \frac{1}{2},\, M^{(i)} = \pm\frac{1}{2}\,\rangle\, \}$

i) Start with $|\, j = 1,\, M = 1\rangle = |\frac{1}{2},\, \frac{1}{2}\rangle\, |\frac{1}{2},\, \frac{1}{2}\rangle$

ii) Successively apply J_- to get to all other states

$$
\begin{aligned}
|1,\, 0\rangle &= \frac{1}{\sqrt{2}}\Big(\, |\tfrac{1}{2},\, -\tfrac{1}{2}\rangle\, |\tfrac{1}{2},\, \tfrac{1}{2}\rangle + |\tfrac{1}{2},\, \tfrac{1}{2}\rangle\, |\tfrac{1}{2},\, -\tfrac{1}{2}\rangle\Big) \\
|1,\, -1\rangle &= |\tfrac{1}{2},\, -\tfrac{1}{2}\rangle\, |\tfrac{1}{2},\, -\tfrac{1}{2}\rangle
\end{aligned}
\tag{2.57}
$$

iii) Find the orthogonal combination to $|\, j_{max},\, M = j_{max} - 1\rangle$:

$$
|0,\, 0\rangle = \frac{1}{\sqrt{2}}\Big(\, |\tfrac{1}{2},\, -\tfrac{1}{2}\rangle\, |\tfrac{1}{2},\, \tfrac{1}{2}\rangle - |\tfrac{1}{2},\, \tfrac{1}{2}\rangle\, |\tfrac{1}{2},\, -\tfrac{1}{2}\rangle\Big) \tag{2.58}
$$

- Rules for coupling $SU(2)$ representations

$j = 0$	$[\,1\,]$	Singlet	$\frac{1}{2} \otimes \frac{1}{2}$: $[\,2\,] \otimes [\,2\,] = [\,1\,] \oplus [\,3\,]$
$j = \frac{1}{2}$	$[\,2\,]$	Doublet	$\frac{1}{2} \otimes 1$: $[\,2\,] \otimes [\,3\,] = [\,2\,] \oplus [\,4\,]$
$j = 1$	$[\,3\,]$	Triplet	$1 \otimes 1$: $[\,3\,] \otimes [\,3\,] = [\,1\,] \oplus [\,3\,] \oplus [\,5\,]$
$j = \frac{3}{2}$	$[\,4\,]$	Quartet	$\vdots$
$\vdots$			
j	$[\,2j + 1\,]$	Multiplet	

- Graphical illustration in terms of weight diagrams:

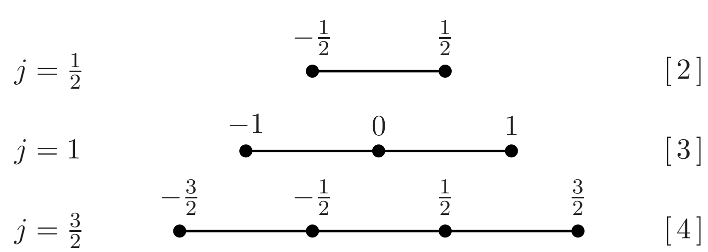
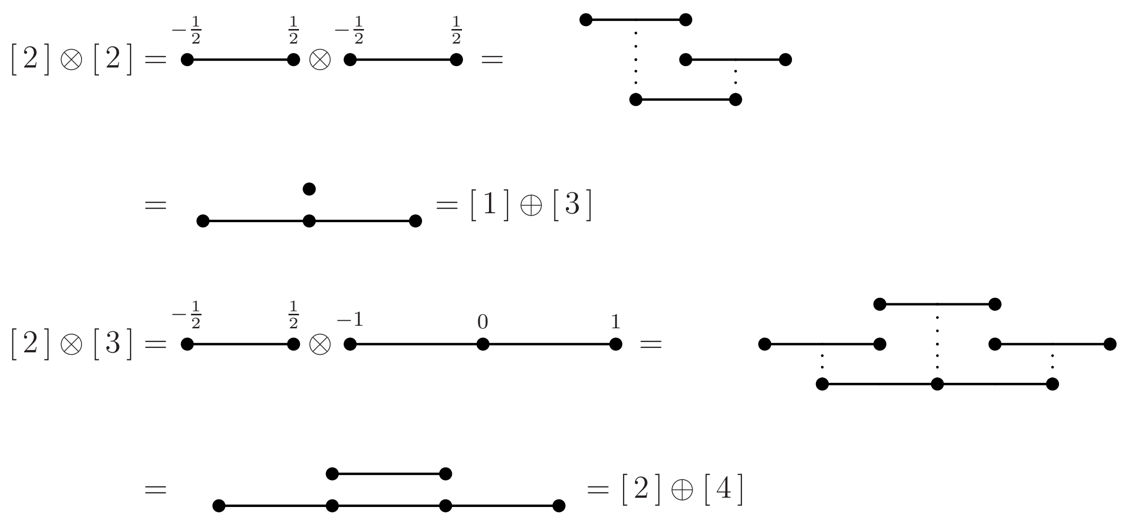


FIG. 2.1: Graphical representation of $SU(2)$ multiplets.

- Building product representations in terms of weight diagrams



- Irreducible representations of $SU(3)$ group: $U = \exp[i\alpha_a t_a]$

$$
t_a = \frac{\lambda_a}{2} \qquad (a = 1, \cdots, 8) \tag{2.59}
$$

– Lie-algebra

$$
\big[t_a\,,\, t_b\big] = i\, f_{abc}\, t_c \tag{2.60}
$$

where f_{abc} is the structure constants of $SU(3)$.

– Anticommutation relations:

$$
\big\{t_a\,,\, t_b\big\} = \frac{1}{3}\,\delta_{ab} + d_{abc}\, t_c \tag{2.61}
$$

where d_{abc} is called "symmetric" structure constants of $SU(3)$.

– Casimir operator in $SU(3)$:

$$
\begin{aligned}
C &= \sum_{a=1}^{8} t_a^2 \\
\left.\begin{aligned} T^2 &= \sum_{i=1}^{3} t_i^2 \\ T_3 &= t_3 \end{aligned}\right\} &\ \text{Isospin} \\
\left. Y = \frac{2}{\sqrt{3}}\, t_8 \right\} &\ \text{Hypercharge}
\end{aligned}
\tag{2.62}
$$

– Raising and lowering operators:

$$
\underbrace{T_\pm = t_1 \pm i\, t_2}_{Iso-spin} \quad , \quad \underbrace{U_\pm = t_6 \pm i\, t_7}_{U-spin} \quad , \quad \underbrace{V_\pm = t_4 \pm i\, t_5}_{V-spin} \tag{2.63}
$$

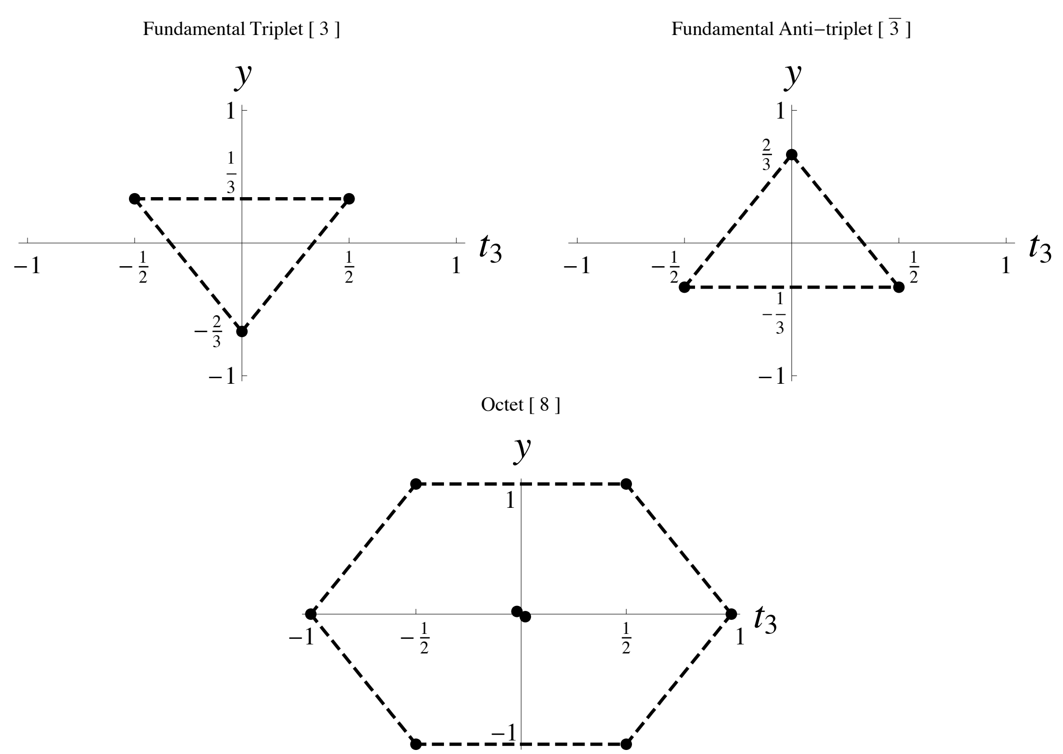
– $SU(3)$ commutation relations:

$$
\begin{aligned}
&[T_3\,,\,T_\pm] = \pm T_\pm && [Y\,,\,T_\pm] = 0 \\
&[T_3\,,\,U_\pm] = \mp\frac{1}{2}U_\pm && [Y\,,\,U_\pm] = \pm U_\pm \\
&[T_3\,,\,V_\pm] = \pm\frac{1}{2}V_\pm && [Y\,,\,V_\pm] = \pm V_\pm
\end{aligned}
\tag{2.64}
$$

$$
\begin{aligned}
[T_+\,,\,T_-] &= 2\,T_3 \\
[U_+\,,\,U_-] &= \frac{3}{2}Y - T_3 \equiv 2\,U_3 \\
[V_+\,,\,V_-] &= \frac{3}{2}Y + T_3 \equiv 2\,V_3
\end{aligned}
\tag{2.65}
$$

$$
\begin{aligned}
&[T_+\,,\,V_+] = [T_+\,,\,U_-] = [U_+\,,\,V_+] = 0 \\
&[T_+\,,\,V_-] = -U_- \qquad [U_+\,,\,V_-] = T_- \\
&[T_+\,,\,U_+] = V_+ \qquad [T_3\,,\,Y] = 0
\end{aligned}
\tag{2.66}
$$

- Weight diagrams of irreducible representations of $SU(3)$



- Product representations and Clebsch-Gordan coefficients of $SU(3)$

– Basis states: $\big|\,[\alpha]\;t\,,\,t_3\,,\,y\,\big\rangle$,

where $[\alpha]$ denote representations e.g., $[3]$, $[8]$ etc.

– 1st step :

$$
\left|\begin{matrix} T\,,\;T_3 \\ [\alpha]\,t\,y\,,\;[\beta]\,t'\,y' \end{matrix}\right\rangle = \sum_{t_3 t_3'} \Big(t\,t_3\,t'\,t_3'\big|T\,T_3\Big)\,\big|\,[\alpha]\;t\,,\,t_3\,,\,y\big\rangle\big|\,[\beta]\;t'\,,\,t_3'\,,\,y'\,\big\rangle \tag{2.67}
$$

– 2nd step:

$$
\big|\,[\gamma]\;T\,,\,T_3\,,\,Y\,\big\rangle = \sum_{t\,y\,t'\,y'} \underbrace{\left(\begin{matrix}[\alpha]\;t\;y \\ [\beta]\,t'\,y'\end{matrix}\middle|\,[\gamma]\,T\,Y\right)}_{\text{Isoscalar } SU(3) \text{ factors}} \left|\begin{matrix} T\,,\;T_3 \\ [\alpha]\,t\,y\,,\;[\beta]\,t'\,y' \end{matrix}\right\rangle \tag{2.68}
$$

- Product representations and rules in terms of weight diagrams:

  Take "center of gravity" of one representation and place it on all parts of the second representation

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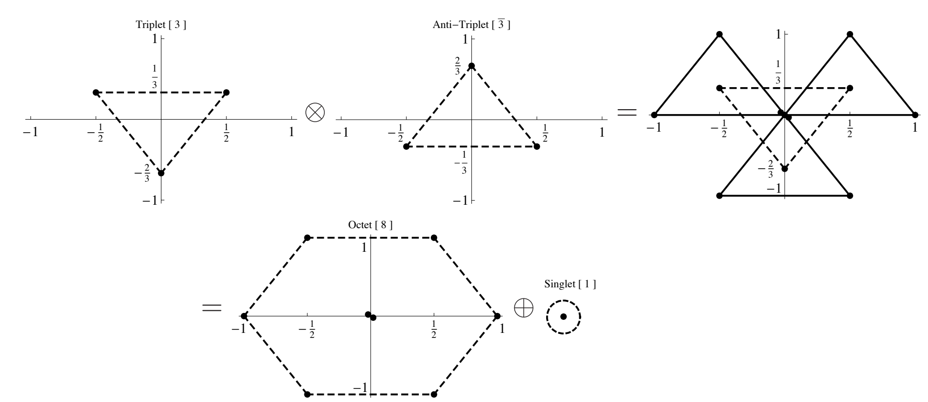
Example. $[3]\otimes[\bar{3}] = [8]\oplus[1]$



- Eigenvalues of Casimir operators

$$
C = \sum_{a=1}^{8} t_a^2 = \frac{1}{4}\sum_{a=1}^{8}\lambda_a^2 = \vec{t}^{\,2} = \frac{1}{4}\vec{\lambda}^2 \tag{2.69}
$$

Representations		Eigenvalues of C
Singlet	$[1]$	0
Triplet	$[3]$	$\frac{4}{3}$
Anti-triplet	$[\bar{3}]$	$\frac{4}{3}$
Sextet	$[6]$	$\frac{10}{3}$
Octet	$[8]$	3