Elementary Particle Dynamics (1)
Quantum Electrodynamics (QED)

From Schrödinger to Dirac

Dirac: from Disaster to Triumph

QED through local gauge invariance

Getting a feeling for calculating Feynman diagrams

Two classic experiments : the power of QED

The Fundamental Forces

Presently: we see four forces in nature

Force	Strength*	Theory	Mediator
Strong	10	Chromodynamics (QCD)	Gluon
Electromagnetic	10^{-2}	Electrodynamics (QED)	Photon
Weak	10^{-13}	(Flavordynamics) Glashow-Weinberg-Salam	W, Z
Gravitational	10^{-42}	General Theory of Relativity	Graviton

- Strength: to be taken as an indication; depends on force, energy, distance (and maybe on time !)

From Schrödinger to Dirac Equation

- Schrödinger equation: non-relativistic quantum-mechanical description
- Heuristic way to 'derive' it
  - from classical energy-momentum relation $\frac{\vec{P}^2}{2m} + V = E$
  - applying the quantum prescription $\vec{p} \to i\hbar\nabla, \quad E \to i\hbar\frac{\partial}{\partial t}$
  - with resulting operators acting on 'wave function' Ψ
- $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$ Schrödinger equation
- One possible relativistic generalization is Klein-Gordon equation, describing particles with spin = 0
  - starting with relativistic energy-momentum relation

$$E^2 - \vec{p}^2 c^2 = m^2 c^2 \quad \textit{or better} \quad p^\mu p_\mu - m^2 c^2 = 0$$

- $-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$ Klein-Gordon equation

Dirac Equation

- Schrödinger derived initially the Klein-Gordon equation, but realized that it
  - does not reproduce energy levels for hydrogen (K-G applies to spin 0)
  - is not compatible with Born's statistical interpretation
    - $|\psi(\vec{r})|^2$… probability of finding particle at point $\bar{r}$
    - this problem can be traced to fact that K-G is second order in t (time)

- 1934: Pauli and Weisskopf showed that statistical interpretation must be reformulated in relativistic quantum theory⇒ relativistic theory must account for pair production and annihilation ⇒ number of particles is not conserved ⇒ showed that Klein-Gordon equation is appropriate for spin = 0 particles

- Dirac: aimed to find equation, consistent with relativistic energy-momentum formula and first order in time

Dirac's Approach

- Strategy: 'factorize' energy-momentum relation $p^\mu p_\mu - m^2c^2 = 0$
  - easy if $\vec{p} = 0$ $\quad (p^0)^2 - m^2c^2 = (p^0 + mc)(p^0 - mc) = 0$

- but with spatial components included, need something like

$$p^\mu p_\mu - m^2c^2 = (\beta^\kappa p_\kappa + mc)(\gamma^\lambda p_\lambda - mc) = \beta^\kappa \gamma^\lambda p_\kappa p_\lambda - mc(\beta^\kappa - \gamma^\kappa)p_\kappa - m^2c^2$$

- or explicitly: $\quad (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 - m^2c^2 =$

$$= (\beta^0 p^0 - \beta^1 p^1 - \beta^2 p^2 - \beta^3 p^3 + mc)(\gamma^0 p^0 - \gamma^1 p^1 - \gamma^2 p^2 - \gamma^3 p^3 - mc)$$

- this gives 8 coefficients to be determined; to reach our goal:
- must avoid terms linear in p_κ , required that $\beta^\kappa = \gamma^\kappa$;
- and finally need to find γ^κ such that $p^\mu p_\mu = \gamma^\kappa \gamma^\lambda p_\kappa p_\lambda$

$p^{\mu} p_{\mu} = \gamma^{\kappa} \gamma^{\lambda} p_{\kappa} p_{\lambda}$ written out explicitly

$$(p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = (\gamma^0)^2 (p^0)^2 + (\gamma^1)^2 (p^1)^2 + (\gamma^2)^2 (p^2)^2 + (\gamma^3)^2 (p^3)^2$$

$$+ (\gamma^0 \gamma^1 + \gamma^1 \gamma^0) p_0 p_1 + (\gamma^0 \gamma^2 + \gamma^2 \gamma^0) p_0 p_2 + (\gamma^0 \gamma^3 + \gamma^3 \gamma^0) p_0 p_3$$

$$+ (\gamma^1 \gamma^2 + \gamma^2 \gamma^1) p_1 p_2 + (\gamma^1 \gamma^3 + \gamma^3 \gamma^1) p_1 p_3$$

$$+ (\gamma^2 \gamma^3 + \gamma^3 \gamma^2) p_2 p_3$$

Dirac's Stroke of a Genius

- As long as the coefficients γ^μ are numbers ⇒ impossible to avoid cross terms such as $\gamma_1\, \gamma_3\, p_1\, p_3$,…
- Dirac's brilliant idea: what if γ's are not numbers, but matrices ?
  - matrices do not commute ⇒ should be possible to find
  - $(\gamma^0)^2 = (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$

    $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \quad \text{for} \quad \mu \neq \nu$
  - or more succinctly $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
  - $g^{\mu\nu}$… Minkowski metric (4*4 matrix with 1, -1,-1,-1) in diagonal, rest=0) ; { } denotes anticommutator {A,B} = AB+BA
- Smallest matrices that work are 4 x 4; among the number of equivalent sets: 'Bjorken and Drell' convention most frequently used

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

σ^i …..Pauli matrices

1 denotes 2 x 2 unit matrix

Dirac Equation

- As a 4 x 4 matrix equation, relativistic energy momentum relation does factor

$$\left(p^{\mu} p_{\mu} - m^2 c^2\right) = \left(\gamma^{\kappa} p_{\kappa} + mc\right)\left(\gamma^{\lambda} p_{\lambda} - mc\right) = 0$$

- Choose one of the two factors: conventional choice

$$\gamma^{\mu} p_{\mu} - mc = 0 \qquad p_{\mu} \rightarrow i\hbar\partial_{\mu}$$

- $$i\hbar\gamma^{\mu}\partial_{\mu}\psi - mc\,\psi = 0 \qquad \textit{Dirac equation}$$

  – Ψ is a four-element column matrix

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} \qquad \text{Dirac} - \text{Spinor}$$

Solution to Dirac Equation: Disaster turned into triumph

- Assume ψ is independent of position

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0 \textit{ describes state with } \vec{p} = 0 \ (\textit{particle at rest})$$

- Dirac equation reduces to: $\frac{i\hbar}{c}\gamma^0 \frac{\partial\psi}{\partial t} - mc\psi = 0$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A / \partial t \\ \partial\psi_B / \partial t \end{pmatrix} = -i\frac{mc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\textit{upper two components}: \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \ \textit{lower two components}: \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

Solution to Dirac Equation: Disaster turned into triumph

$$\frac{\partial \psi_A}{\partial t} = -i\left(\frac{mc^2}{\hbar}\right)\psi_A; \quad -\left(\frac{\partial \psi_B}{\partial t}\right) = -i\left(\frac{mc^2}{\hbar}\right)\psi_B$$

- solutions

$$\psi_A(t) = e^{-i\left(mc^2/\hbar\right)t}\psi_A(0); \psi_B{}^{(t)} = e^{+i\left(mc^2/\hbar\right)t}\psi_B{}^{(0)}$$

$e^{-iEt/\hbar}$ … time dependence of quantum state with energy $E = mc^2$ (particle at rest)

- ψ_A corresponds to state with **p** = 0, as expected
- ψ_B = ? state with negative energy ($E = -mc^2$) : the famous ‘disaster’
- ψ_B Dirac’s way out: unseen ‘sea’ of negative-energy particle
- Pauli et al: particles describes antiparticle with positive energy

Dirac Equation with **p** = 0

- Dirac equation with **p** = 0 has four independent solutions
- $$\psi^{(1)} = e^{-i\left(mc^2/\hbar\right)t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad \psi^{(2)} = e^{-i\left(mc^2/\hbar\right)t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$ electron spin up; spin down

  $$\psi^{(3)} = e^{+i\left(mc^2/\hbar\right)t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\quad \psi^{(4)} = e^{+i\left(mc^2/\hbar\right)t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$ positron spin up; spin down

Dirac Equation: Plane wave solution

- Next step: plane-wave solution $\psi(x) = a e^{-ikx} u(k)$
  - describes particle with specified energy and momentum
  - find four-vector k^μ and associated bispinor $u^{(k)}$ such that $\psi(x)$ satifies the Dirac equation; putting this into Dirac equation and…
  - after several pages of matrix manipulation ….

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \quad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ \frac{c(-p_z)}{E+mc^2} \end{pmatrix}; \quad v^{(1)}; v^{(2)}$$

  - customary to use *v* for antiparticle (instead of *u);* $N=((E=mc^2)/c)^{1/2}$

Conceptual Next Steps

- u ….. are the particles, satisfying ($\gamma^u p_u$ – mc) u = 0;
  ν ….. are the antiparticles (($\gamma^u p_\mu$ + mc) ν = 0)
- $u^{(1)}$ is electron with spin up, $u^{(2)}$ electron with spin down
- Similar development for photons; example for plane wave:

$$A_\mu(x) = a e^{-(i/\hbar)px} \varepsilon_\mu^{(s)}, \quad s = 1,2$$ for the two spin (polarization) states

- In modern language: Lagrangian invariant under local gauge transformation U(1) -> generates gauge field A_μ

Glimpse at Field Theory of QED

- In classical particle mechanics: calculate position as a function of time
- In Field Theory: calculate one or several functions (e.g. temperature, electric potential) as function of position, time: $\phi(x, y, z, t)$
- Classically: Lagrangian $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$
- Field Theory: Lagrangian (density): function of the fields Φ, x,y,z,t

$$\mathcal{L} = \mathcal{L}(\phi_i, \partial_\mu \phi_i) \qquad \partial_\mu \phi_i \equiv \frac{\partial \phi_i}{\partial x^\mu}$$

- Classically, law of motion described by Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}, i = 1,2,3$$

- Relativistic Theory: simplest generalization

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i}; \ i = 1,2,....$$

Dirac Lagrangian for Spinor ψ (S=½) Field

- Consider Dirac Lagrangian for a Spinor (Spin ½) field

$$\mathcal{L} = i(\hbar c)\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - (mc^2)\overline{\psi}\psi$$

- Treating ψ and the adjoint spinor $\overline{\psi}$ as independent field variables and

- applying Euler-Lagrange $i\gamma^{\mu}\partial_{\mu}\psi - \left(\frac{mc}{\hbar}\right)\psi = 0$

  - gives Dirac equation, describing in quantum field theory a particle with spin ½ and mass *m*

- Corresponding 'momentum space' equation

$$\left[p - (mc)\right] = 0$$

  - Corresponding propagator for the free Lagrangian is $i / \left[p - (mc)\right]$

Local Gauge Invariance → Lagrangian of QED

- Dirac Lagrangian is invariant under transformation $\psi \rightarrow e^{i\theta}\psi$ (global phase transformation); θ … any real number
- However, if θ is a function of space-time x^μ
- 'Local' phase transformation: $\psi \rightarrow e^{i\theta(x)}\psi$

⇒ however $\mathcal{L} \rightarrow \mathcal{L} - \hbar c(\partial_\mu \theta)\overline{\psi}\gamma^\mu \psi$; NOT invariant

or with $\lambda(x) = \frac{\hbar c}{q}\theta(x)$ $\mathcal{L} \rightarrow \mathcal{L}(q\overline{\psi}\gamma^\mu \psi)\partial_\mu \lambda$

- New concept: require invariance of $\mathcal{L}$ under local phase transformation, must add extra term $\mathcal{L} = [i\hbar c\overline{\psi}\gamma^\mu \partial_\mu \psi - mc^2\overline{\psi}\psi] - (q\overline{\psi}\gamma^\mu \psi)A_\mu$

with A_μ a new field, such that $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$ (gauge invariance)

- Complete $\mathcal{L}$ includes is invariant at the price of a new term for free field A_μ

$$\mathcal{L} = [i\hbar c\overline{\psi}\gamma^\mu \partial_\mu \psi - mc^2\overline{\psi}\psi] - \left[\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right] - (q\overline{\psi}\gamma^\mu \psi)A_\mu \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Lagragian of QED

Local Gauge Invariance

- Demanding local gauge invariance introduces vector field A^μ; must be massless, because otherwise gauge invariant would be lost
  - ⇒ generates all of the electrodynamics and specifies the current produced by the Dirac particles

- Idea of local gauge invariance introduced by Hermann Weyl in 1918

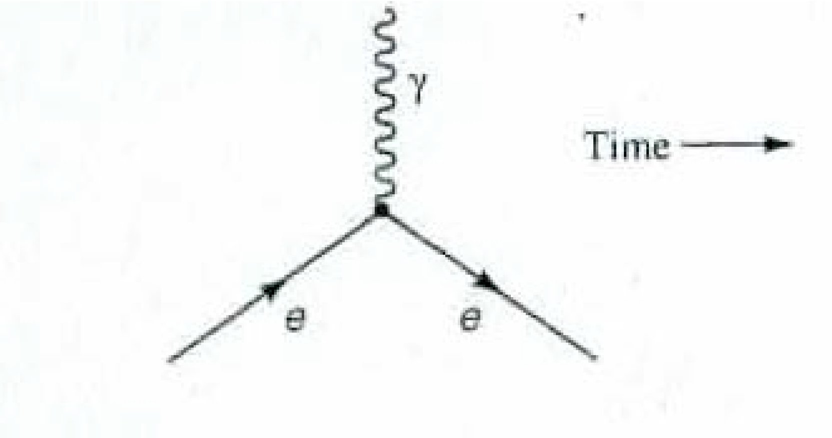
- Its power was not fully appreciated until the early 1970's
- 't Hooft, Veltman: have shown that under certain conditions quantum field theories with local gauge invariance are renormalizable (will be explained later); Nobel Prize in Physics in 1999

From U(1) to SU(2) to SU(3)

- Phase transformation can be considered as $\psi' = U \psi$
- $U = e^{i\theta}$; $U^+ U = 1$
- Group of all such matrices is U(1) ; …. is a 1x1 matrix
- Symmetry involved is called U(1) gauge invariance
- Young and Mills applied it to other field theories: SU(2) ⇒ describes interaction of Dirac fields with three massless vector gauge fields (would be identified later with W^+, W^-, Z^0)
- Idea extended to SU(3), generating QCD
- In Standard Model all of the fundamental interactions are generated through the requirement of local gauge invariance under
  $U(1) \otimes SU(2) \otimes SU(3)$ transformation
- Truely breathtaking: laws of Nature derived with one elegant concept

For ‘General’ Culture: Feynman Rules

- Pictorial ‘code’ to represent particle interactions
- All electromagnetic processes are ultimately reducible to the process represented by the diagram below
  - Convention for interpreting the diagram



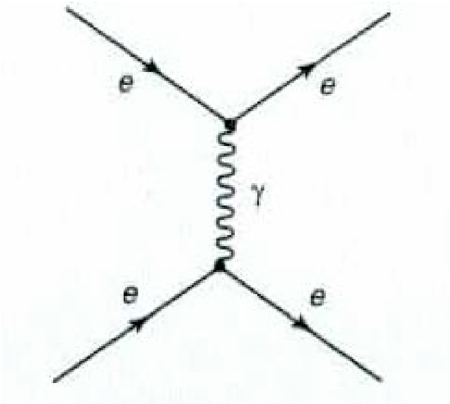
- time flows horizontally
- the charged particle enters
- emits (or absorbs) a photon
- the charged particle exits
- charged particle could be
  - charged lepton
  - a quark

Feynman Diagrams

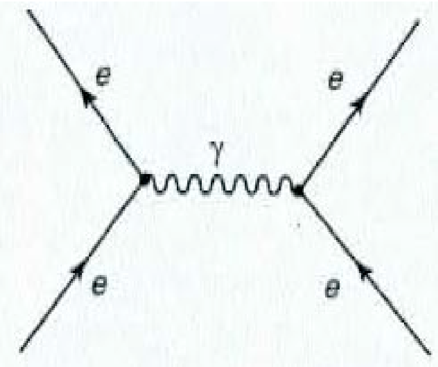
- Feynman diagrams are purely symbolic
- Horizontal dimension represents time
- Vertical dimension does NOT correspond to physical separation
- Quantitatively, each Feynman diagram represents a particular number, which can be calculated -> ‘Feynman rules’
- Approach
  - draw/calculate all the diagrams contributing to a process
  - sum of all Feynman diagrams with the specific external lines represents the physical process
- In principle: an infinite number contribute
- In practice: saved by the fact that fine structure constant $\alpha = e^2 / hc \approx 1/137$
- Higher orders contribute less; need only consider processes up to certain order, consistent with experimental accuracy/ aims/ tests

QED: ‘Feynman diagrams’: Pictorial description + theoretical rules

More complicated processes can be built up with combinations of this ‘primitive’ vertex

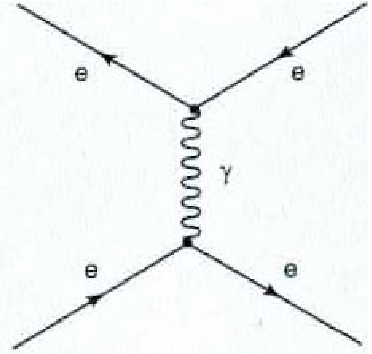


- two electrons enter
- a photon is exchanged between them
- the two electrons exit
- classically: Coulomb repulsion
- in QED: ‘Møller Scattering’



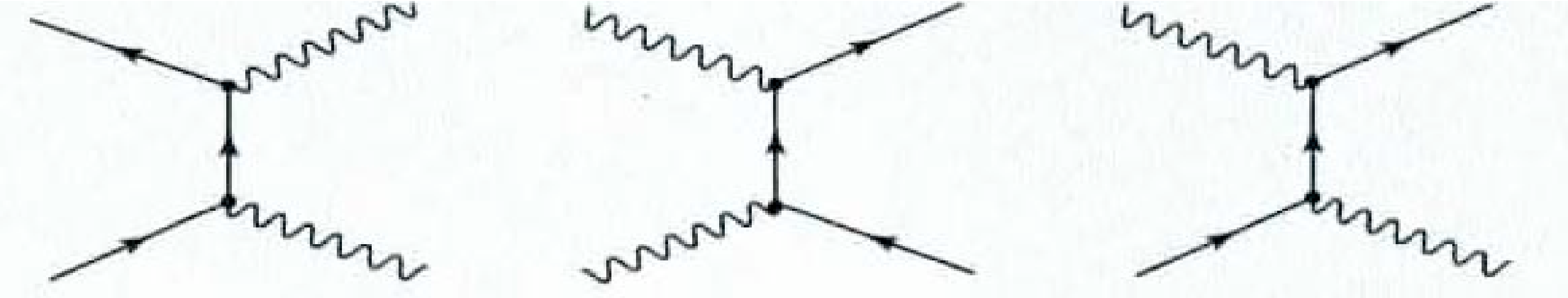
- arrow pointing back in time -> antiparticle going forward in time
- this process represents electron-positron annihilation; photon is formed, which produces electron-positron pair: ‘Bhabha scattering’

QED: More processes



electron-positron scattering: also contributing to 'Bhabha Scattering'

classically: Coulomb attraction

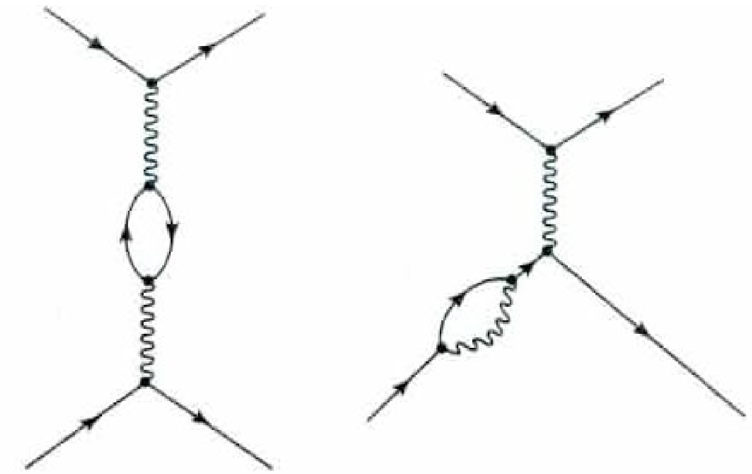


$e^+ + e^- \rightarrow \gamma + \gamma$
(pair annihilation)

$\gamma + \gamma \rightarrow e^+ + e^-$
pair production

$e^- + \gamma \rightarrow e^- + \gamma$
(Compton Scattering)

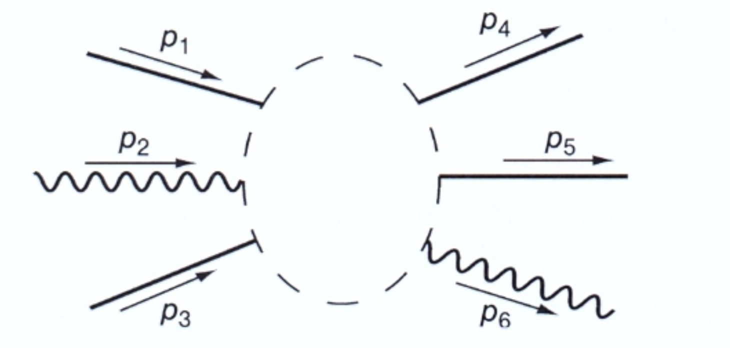
QED: Virtual Particles

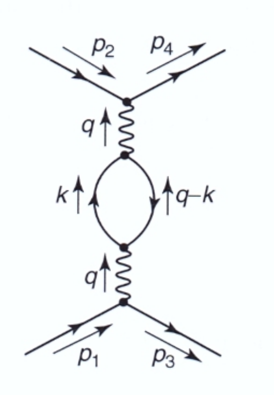


- both diagrams describe 'Møller Scattering'
- the internal lines/diagrams are not observed ('virtual' particles)
  - Virtual particle production allowed due to
    Heisenberg uncertainty relation
- the internal lines describe the mechanism and contribute to the process in measurable ways
- only the external lines are observed

For ‘General’ Culture: Feynman Rules

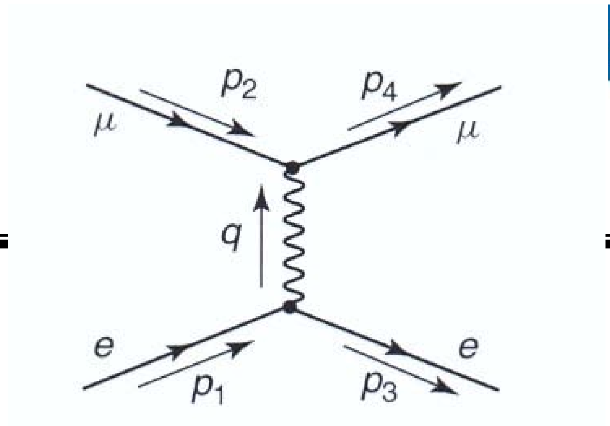
- Notation: see Figure





- Electrons: incoming: u , outgoing: $\bar{u}$ (u spinor)
  Positrons: incoming: $\bar{v}$, outgoing: v (v spinor)
  Photons: incoming: ε_μ , outgoing: ε_μ^*
- Vertex contributes $i g_e \gamma^\mu$ $g_e = \sqrt{4\pi\alpha}$ …coupling constant
- Propagator e^+, e^- $\frac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$ photons $-i g_{uv} / q^2$
  q_μ are internal momenta
- Conservation of energy, momenta $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$
- k_i are the four-momenta coming into the vertex

Example
Electron-muon scattering



- $e + \mu \rightarrow e + \mu$

- Mott scattering for M >> m → Rutherford scattering ν << c

$$(2\pi)^4 \int \left[\bar{u}^{(s_3)}(p_3)\left(ig_e\gamma^{\mu}\right)u^{(s_1)}(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}^{(s_4)}(p4)\left(ig_e\gamma^{\nu}\right)u^{(s_2)}(p2)\right]$$
$$\times\,\delta^4(p_1 - p_3 - q)\ \delta^4\ (p_2 + q - p_4)\,d^4q$$

after q (= internal momenta) integration, amplitude

$$\mathcal{M} \;=\; -\frac{g_e^2}{(p_1 - p_3)}\left[\bar{u}^{(s_3)}(p_3)\,\gamma^{\mu}u^{(s_1)}(p_1)\right]\ \left[\bar{u}^{(s_4)}(p_4)\,\gamma_{\mu}u^{(s_2)}(p2)\right]$$

- looks complicated (four spinors, 8 γ matrices , but this is just a number,which can be calculated, once the spin states are defined

Example
Electron-muon scattering

Calculate the electron-muon scattering amplitude in CM system (electron and muon scatter along z-direction); initial and final particles have helicity +1

First, we need to evaluate the bispinors; for our case:

$p_x = p_y = 0$; $cp_z = c|p| = ((E-mc^2)(E+mc^2))^{1/2}$

$$u^{(1)} = \frac{\sqrt{E+mc^2}}{\sqrt{c}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{\frac{(E-mc^2)}{(E+mc^2)}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{c}} \begin{pmatrix} \sqrt{E+mc^2} \\ 0 \\ \sqrt{E-mc^2} \\ 0 \end{pmatrix}$$

Example
Electron-muon scattering

$$u^{(2)} = \frac{1}{\sqrt{c}} \begin{pmatrix} 0 \\ \sqrt{E+mc^2} \\ 0 \\ -\sqrt{E-mc^2} \end{pmatrix}$$

$$\nu^{(1)} = \frac{\sqrt{E+mc^2}}{\sqrt{c}} \begin{pmatrix} 0 \\ \dfrac{-\sqrt{(E-mc^2)(E+mc^2)}}{(E+mc^2)} \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{c}} \begin{pmatrix} 0 \\ -\sqrt{E-mc^2} \\ 0 \\ \sqrt{E+mc^2} \end{pmatrix}$$

$$u^{(2)} = \frac{1}{\sqrt{c}} \begin{pmatrix} \sqrt{E-mc^2} \\ 0 \\ \sqrt{E+mc^2} \\ 0 \end{pmatrix}$$

Example
Electron-muon scattering

For our problem we have specifically:

$a_{+-} = ((E_e \pm mc^2)/c)^{1/2}$; $b_{+-} = ((E_\mu \pm Mc^2)/c)^{1/2}$

$$u(1)=\begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix}, u(2)=\begin{pmatrix} 0 \\ b_+ \\ 0 \\ b_- \end{pmatrix}, u(3)=\begin{pmatrix} 0 \\ a_+ \\ 0 \\ a_- \end{pmatrix}, u(4)=\begin{pmatrix} b_+ \\ 0 \\ b_- \\ 0 \end{pmatrix}$$

$$M = -\frac{g_e^2}{(p_1 - p_3)^2} \left\{ \left[\bar{u}(3)\gamma^0 u(1)\right]\left[\bar{u}(4)\gamma^0 u(2)\right] - \left[\bar{u}(3)\gamma^i u(1)\right]\left[\bar{u}(4)\gamma^i u(2)\right] \right\}$$

where i is summed from 1 to 3

Example
Electron-muon scattering

$$\bar{u}(3)\gamma^0 u(1) = (0\ a_+\ 0\ a_-)\ \gamma^0\gamma^0 \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix} = (0\ a_+\ 0\ a_-) \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix} = 0.$$

$$\bar{u}(3)\gamma^i u(1) = (0\ a_+\ 0\ a_-) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} a_+ \\ 0 \\ a_- \\ 0 \end{pmatrix}$$

$$= (0\ a_+\ 0\ -a_-) \begin{pmatrix} \sigma^i \begin{pmatrix} a_- \\ 0 \end{pmatrix} \\ -\sigma^i \begin{pmatrix} a_+ \\ 0 \end{pmatrix} \end{pmatrix} = (0\ a_+)\ \sigma^i \begin{pmatrix} a_- \\ 0 \end{pmatrix} + (0\ a_-)\ \sigma^i \begin{pmatrix} a_+ \\ 0 \end{pmatrix}$$

$$= 2a_+a_- \left[(0\ 1) \begin{pmatrix} \sigma^i_{11} & \sigma^i_{12} \\ \sigma^i_{21} & \sigma^i_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] = 2a_+a_- \left[(0\ 1) \begin{pmatrix} \sigma^i_{11} \\ \sigma^i_{21} \end{pmatrix} \right] = 2a_+a_-\sigma^i_{21}.$$

Example
Electron-muon scattering

$$\bar{u}(4)\gamma^i u(2) = (b_+ \; 0 \; b_- \; 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & \sigma^i\\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix}0\\ b_+\\ 0\\ b_-\end{pmatrix}$$

$$=(b_+ 0 - b_- 0)\begin{pmatrix}\sigma^i & \begin{pmatrix}0\\ b_-\end{pmatrix}\\ -\sigma^i & \begin{pmatrix}0\\ b_+\end{pmatrix}\end{pmatrix} = (b_+ 0)\sigma^i\begin{pmatrix}0\\ b_-\end{pmatrix} + (b_- 0)\sigma^i\begin{pmatrix}0\\ b_+\end{pmatrix}$$

$$= 2\, b_+ b_- \left[(1 \; 0)\begin{pmatrix}\sigma^i_{11} & \sigma^i_{12}\\ \sigma^i_{21} & \sigma^i_{22}\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}\right] = 2\, b_+ b_- \left[(1 \; 0)\begin{pmatrix}\sigma^i_{12}\\ \sigma^i_{22}\end{pmatrix}\right] = 2\, b_+ b_- \sigma^i_{12} \; .$$

Example
Electron-muon scattering

$$M = -\frac{g_e^2}{(p_1-p_3)^2}(2a_+a_-2b_+b_-)\sigma_{21}\bullet\sigma_{12} = \frac{8g_e^2}{(p_1-p_3)^2}(a_+a_-)(b_+b_-),$$

where $\sigma_{21}\bullet\sigma_{12} = (1)(1)+(i)(-i)+(0)(0) = 2$ in the last step.

Now $(a_+a_-) = \sqrt{\frac{E_e^2-m^2c^4}{c^2}} = \sqrt{\frac{\mathbf{p}_e^2c^2}{c^2}} = |\mathbf{p}_e|$, $(b_+b_-) = |\mathbf{p}_\mu|$, and $|\mathbf{p}_e| = |\mathbf{p}_\mu|$.

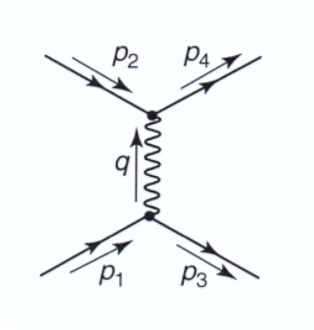
So $M = \frac{8g_e^2\,\mathbf{p}_e^2}{(p_1-p_3)^2}$.

$p_1 = \left(\frac{E_e}{c}, \mathbf{p}_e\right)$, $p_3 = \left(\frac{E_e}{c}, -\mathbf{p}_e\right)$; so $(p_1-p_3) = (0, 2\mathbf{p}_e)$, $(p_1-p_3)^2 = 0-4\mathbf{p}_e^2$.

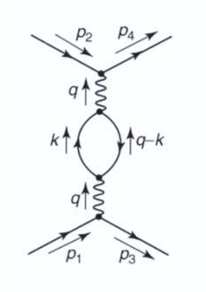
$$\therefore \mathcal{M} = \frac{8g_e^2\,\mathbf{p}_e^2}{-4\mathbf{p}_e^2} = -2g_e^2.$$

Need another new concept: Renormalization

- Electron-muon scattering
  - lowest order diagram:



  - next order correction:

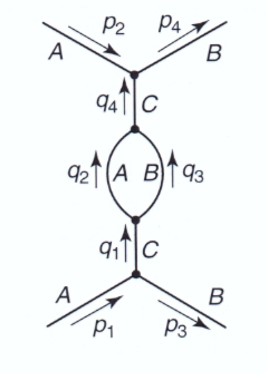


- Next order corrections lead to modification of photon propagator

$$\frac{g_{\mu\nu}}{q^2} \rightarrow \frac{g_{\mu\nu}}{q^2} - \frac{i}{q^4} I_{\mu\nu}$$ and gives divergent integrals

Including Higher Order Contributions

- Applying these rules to diagrams of the form



  leads to expressions of $\int^{\infty} \frac{1}{q^4} q^3 dq \Rightarrow \ell n q \int^{\infty} \to \infty$

  logarithmically divergent at large q

- Twenty year long struggle by some of the greatest physicists: Dirac, Pauli, Kramers, Weisskopf, Bethe, Tomonaga, Schwinger, Feynman … to develop a systematic approach to deal with these infinities to obtain calculable results which could be compared to measurements

Self-Interaction in Classical Physics

- Classical electrodynamics of point particles
  - Electrostatic energy of point charge is infinite, makes infinite contribution to the particle's mass; electrostatic energy required to assemble sphere with charge e and 'effective' radius r_e -> E= mc^2 -> defines classical electron radius

$$m_{em} = q^2 / 8\pi r_e; \quad r_e = \frac{1}{\alpha}\frac{\hbar}{m_e c} \sim 2.8 \times 10^{-15} m$$

r_e (classical electron radius) : mass m_e due to its electrostatic potential energy

- Total effective mass includes the bare mass of the spherical particle in addition to mass associated with field
  - assume, bare mass is allowed to be negative →
  - perhaps possible to take a consistent point limit
  - called 'renormalization' by Lorentz
  - inspiration for later work = renormalization in QFT

- Maybe this is telling us that there are no point particles in nature; point particles only a theoretical construct

Divergences in QED

- Treatment occupied some of the best physicists of the last century: Dirac, Born, Heisenberg, Pauli, Weisskopf, Schwinger, Tomonaga, Feynman, Dyson, ….
- Divergences appear in diagram with closed loops of virtual particles
- Virtual particles may have a mass different from their physical mass: 'off-shell'
- Integrals over the loop processes are often divergent
  - 'ultraviolet' (UV) divergences: loop particles with large momenta
    - short-distance, short-time phenomena
  - 'infrared' divergences: due to massless particles, like photons
    - treated in analogy to bremsstrahlung

Handling of Divergences

- Integrals are of the type

$$\int_{m^2}^{\infty} \frac{d_Z}{Z} \rightarrow \textit{are considered in the form} \int_{m^2}^{M} \frac{d_Z}{Z} = \ell n \frac{M^2}{m^2}$$

- Amplitude $\mathcal{M} = -g_e^2 \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \frac{g_{\mu\nu}}{q^2} \left\{1 - \frac{g_e^2}{12\pi^2}\left[\ln\left(\frac{M^2}{m^2}\right) - f\left(\frac{-q^2}{m^2c^2}\right)\right]\right\}$

$$\times \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$$

- Redefine coupling constant $g_R \equiv g_e \sqrt{1 - \frac{g_e^2}{12\pi^2} \ell n\left(\frac{M^2}{m^2}\right)}$

- Resulting in $\mathcal{M} = -g_R^2 \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \frac{g_{\mu\nu}}{q^2} \left\{1 + \frac{g_R^2}{12\pi^2} f\left(\frac{-q^2}{m^2c^2}\right)\right\} \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right]$

Concept

- Reference to cut off is absorbed in coupling constant

- g_R reflects the actual measurement; we are not measuring the 'bare' charge, but the physical charge, which includes the higher order terms

- Finite correction terms remain, depending on $q^2 \Rightarrow$ coupling depends on q^2

$$g_R(q^2) = g_R(0)\sqrt{1 + \frac{g_R(0)^2}{12\pi^2} f\left(\frac{-q^2}{m^2c^2}\right)}$$

- In terms of $g_e = \sqrt{4\pi\alpha}$ $\quad \alpha(q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi} f\left(\frac{-q^2}{m^2c^2}\right)\right\}$

Regularization of Divergent Integrals

- Regularization: mathematical procedure to cancel divergencies
- Introduce a cut-off procedure
  - introduce factor $\frac{M^2c^2}{q^2 - M^2c^2}$ under integral; M very large
- Integrals can be calculated and seperated into part independent of M; second term depending logarithmically on M
- With a surprising result: all M-dependent terms appear in the final answer in the form of
  - addition to the masses and the couplings
  - $m_{physical} = m + \delta m$ ($\rightarrow \infty$ for $M \rightarrow \infty$)
  - $g_{physical} = g + \delta g$ ($\rightarrow \infty$ for $M \rightarrow \infty$)
- Modern approach is Lorentz-invariant ‘Dimensional Regularization’:
- Four dimensions replaced with 4D-ε: result is a convergent part and part divergent as 1/ ε

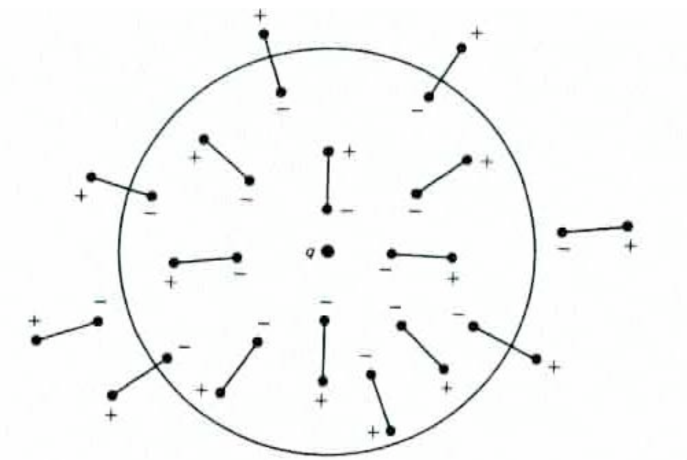
Renormalization

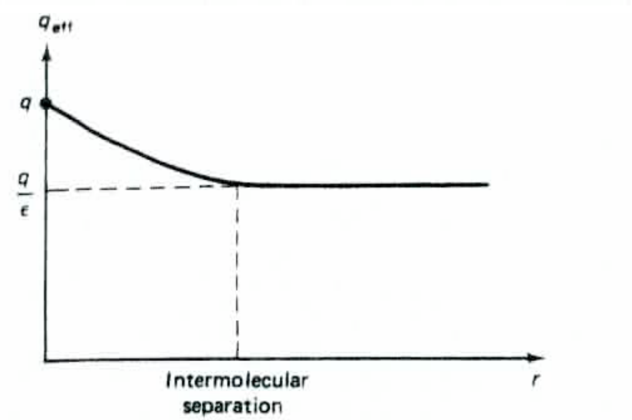
- Insight: quantities appearing in the Lagrangian (mass, charge, coupling strength) do not correspond to the physical constants measured
- ‘Bare’ quantities do not take into account contributions of virtual particle loop effects, which contribute to the physical constants
- Formulae have to be rewritten in terms of measureable, renormalized quantities → renormalization scale, which is characteristic to a specific measurement
- Example: charge of an electron would be defined as a quantity at the renormalization scale
- This procedure introduces the concept of the ‘Running coupling constants’ → describes the changing behaviour of the QFT under change of the energies involved
- Conceptual example: $I = \int_0^a \frac{1}{z}\,dz - \int_0^b \frac{1}{z}\,dz = \ell n a - \ell n 0 - \ell n b - \ell n 0$ ill defined

take lower limit $\varepsilon_a, \varepsilon_b : I = \ell n \frac{a}{b} - \ell n \varepsilon_A + \ell n \varepsilon_B \rightarrow \ell n \frac{a}{b}$ $for\ \varepsilon_a, \varepsilon_b \rightarrow 0$

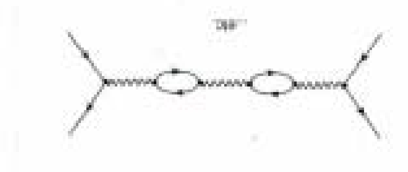
Running coupling constant in QED

- Also in electrodynamics: effective coupling also depends on distance
  - Charge q embedded in dielectric medium ε (polarizable)





- in QED: vacuum behaves like dielectric
- full of virtual positron-electron pairs



- virtual electron attracted to q, positron repelled
- medium becomes polarized
- Particle q acquires halo of negative particles, partially screening the charge q
- at large distance charge is reduced to q / ε
- vacuum polarization screens partially the charge at distances larger than h/mc= $2.4 \cdot 10^{-10}$ cm (Compton wavelength of electron)
- Measurable, e.g. in structure of hydrogen levels
- NOTE: we measure the 'screened' charge, not the 'bare' charge

Running Coupling Constants

- Effective charge of electron (muon) depends on momentum transferred, i.e. on distance of approach ⇒ consequence of vacuum polarization, which ‘screens’ the charge

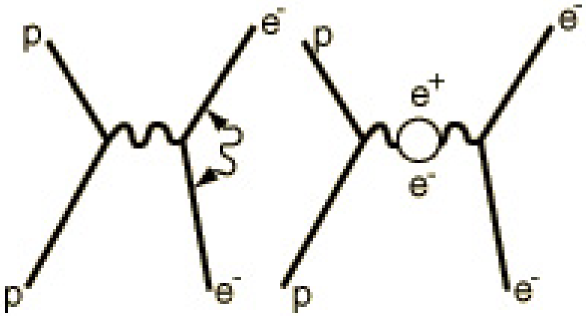
- Effect only significant at high energies
  - At head-on collision at v = 0.1c ⇒
  - effect is at level of ~ 6 x 10^{-6}

- However, as Lamb shift measurement shows, it is detectable; also directly measured in e^+e^- - collisions

Lamb Shift in Hydrogen

- Hydrogen levels calculated with Dirac equation
  - $^2S_{1/2}$ and $^2P_{1/2}$ levels have precisely the same energy (are ‘degenerated’)

- However, in QED we have additional diagrams



self interaction, vacuum polarization

- Self interaction ‘smears’ position of electron over a range of
  - ~ 0,1 fermi (Bohr radius is 52900 fermi)
  - weakening the force on S-electron (which approaches nucleus closer) more than $^2P_{1/2}$ electron
  - $^2S_{1/2}$ level is ~ 4.3 x 10^{-6} eV above $^2P_{1/2}$ level
  - $\Delta E_{Lamb} \sim \alpha^5 f(n, l, j)$

Lamb-Retherford Experiment

- Need to form a beam of metastable 2 $^2S_{1/2}$ states
- Induce microwave transition between $^2S_{1/2}$ and $^2P_{1/2}$, which decays in ~10^{-9} sec under emission of light
- Hydrogen produced in tungsten oven → bombarded by electrons → to excite $^2S_{1/2}$ states(1 in ~10^8 !) → impinge on metal plate, where they eject electrons and can therefore be detected
- Radio frequency transition from $^2S_{1/2}$ to $^2P_{1/2}$ states quenches $^2P_{1/2}$ states
- Transition frequency is $f \approx 1054$ MHz

Apparatus of Lamb and Retherford

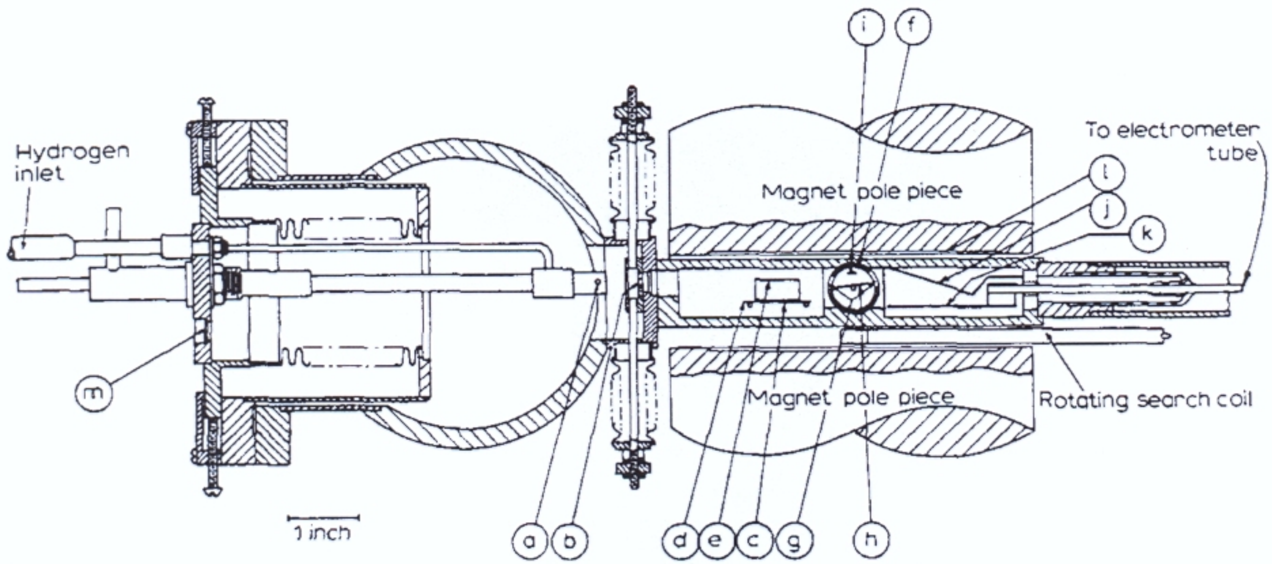


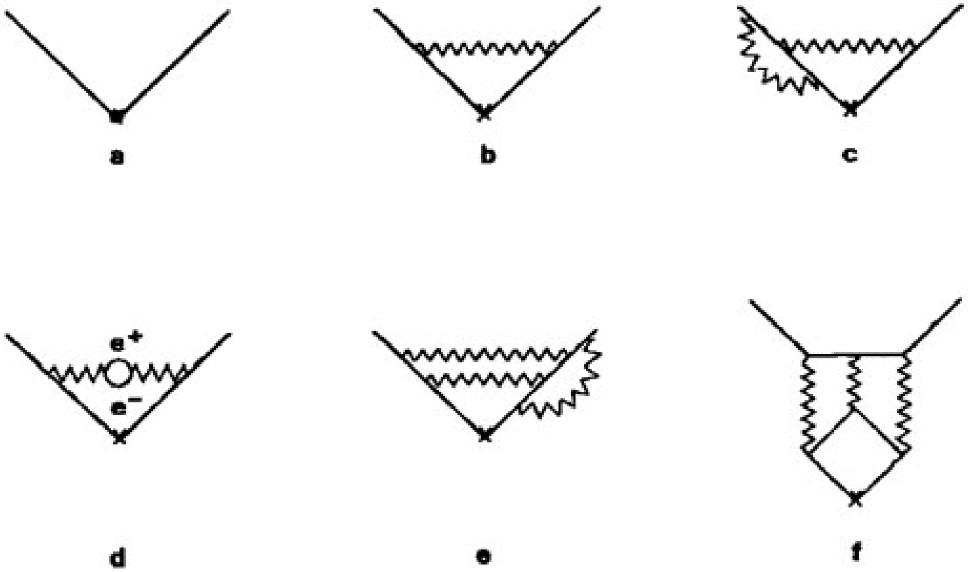
Fig. 3. Cross section of second apparatus: *(a)* tungsten oven of hydrogen dissociator, *(b)* movable slits, *(c)* electron bombarder cathode, *(d)* grid, *(e)* anode, *(f)* transmission line, *(g)* slots for passage of metastable atoms through interaction space, *(h)* plate attached to center conductor of r-f transmission line, *(i)* d.c. quenching electrode, *(j)* target for metastable atoms, *(k)* collector for electrons ejected from target, *(l)* pole face of magnet, *(m)* window for observation of tungsten oven temperature.

g-2 of the Muon

- Magnetic moment $\mu = g\frac{e}{2mc} \bullet \left(\frac{\hbar}{2}\right)$
- If 'Dirac' particle: g = 2, exactly
- The value is modified by quantum fluctuations in the field around the muon
  - QED-effects of fluctuations: $\sim 10^{-3}$
  - electroweak effects (virtual W, Z): $\sim 10^{-8}$
  - strong interaction effects: $\sim 10^{-7}$
- Present value for $a_\mu = (g_\mu - 2) / 2 =$

  $= (11659208.0 \pm 6.3) \times 10^{-10}$
- Biggest theoretical uncertainty: hadronic vacuum polarization contrib.
  - determined from e+e- -> hadrons or $\tau \rightarrow$ hadrons
- Δ (Measurement – SM-Theory) ~ 3.36 σ (e^+e^-) 0.96 σ (τ data)
- A genuine difference between Standard Model Theory and experiment would imply 'New Physics' (e.g. Supersymmetry)

Diagrams contributing to anomalous magnetic moment of the muon



- solid line… muon; zig-zag line…photon; closed loops... creation of virtual electron-positron pair

g-2 Precession

- Longitudinally polarized particle, moving in uniform magnetic field B
  - momentum vector turns at cyclotron frequency $f_c = eB/2\pi\ mc$
  - spin precession frequency is the same as for particle at rest:

    $$2\pi f_s = 2\mu\ B/h = g\ (eB/2mc) = (1+a_\mu)\ (eB/1mc)$$

  - if $g = 2 \Rightarrow f_c = f_s$
  - if $g > 2$, spin turns faster than momentum vector
  - in laboratory, rotating frequency of spin relative to momentum vector is

    $$2\pi f_a = 2\pi\ (f_s - f_c) = a_\mu\ (eB/mc)$$

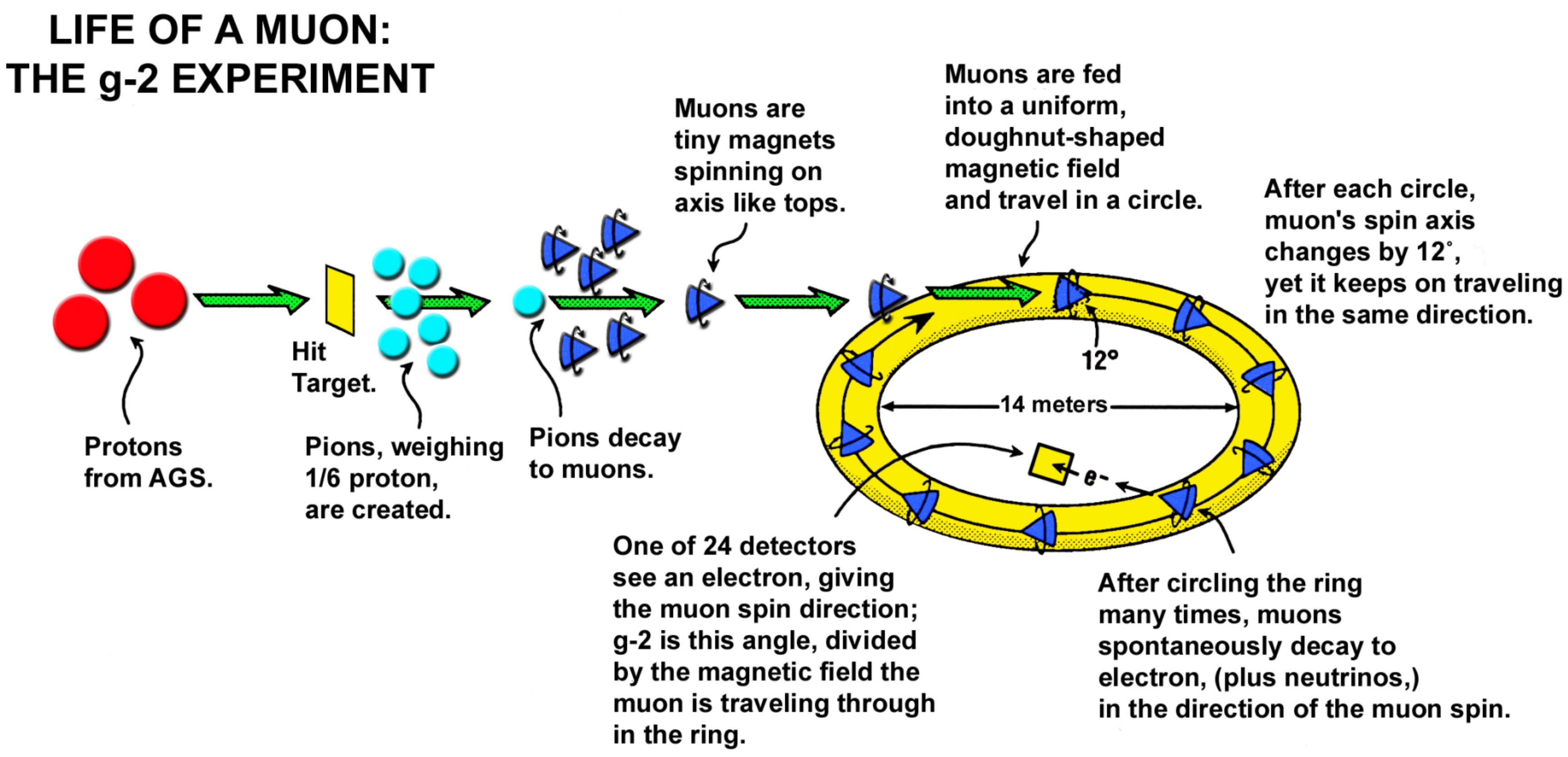
Comments on most recent (g-2) experiment at Brookhaven National Laboratory

- BNL uses continuous magnet, with field known to 0.1 ppm at 1.451 Tesla
- Polarized μ's moving in $\vec{B}$ $\perp$ to muon spin and $\perp$ to plane of the orbit of electric quadrupole field $\vec{E}$ (used for vertical focussing)
- Muons are stored at magic momentum of 3.094 GeV/c in uniform magnetic field → electric fields to focus muons do not disturb muon anomaly measurement
- Frequency difference ω_a between precession frequency ω_s and cyclotron frequency ω_c is

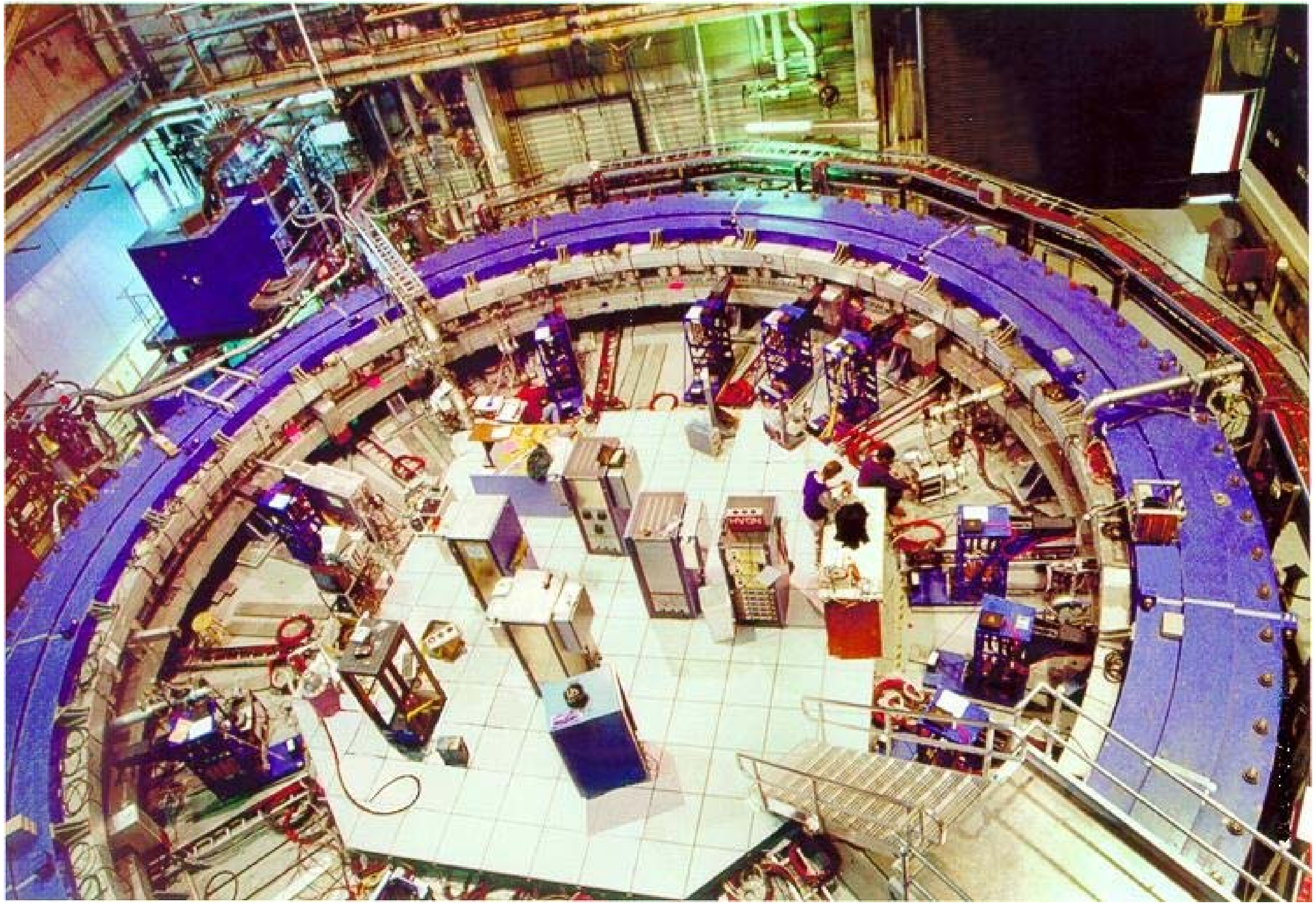
$$\vec{\omega}_a = -\tfrac{e}{m}\left[a_\mu \vec{B} - \left(a_\mu - \tfrac{1}{\gamma^2 - 1}\right)\vec{\beta}\times\vec{E}\right]$$

- No $\vec{E}$ - dependence for γ = 29.3
- Achieved accuracy of 0.35 parts per million (ppm)

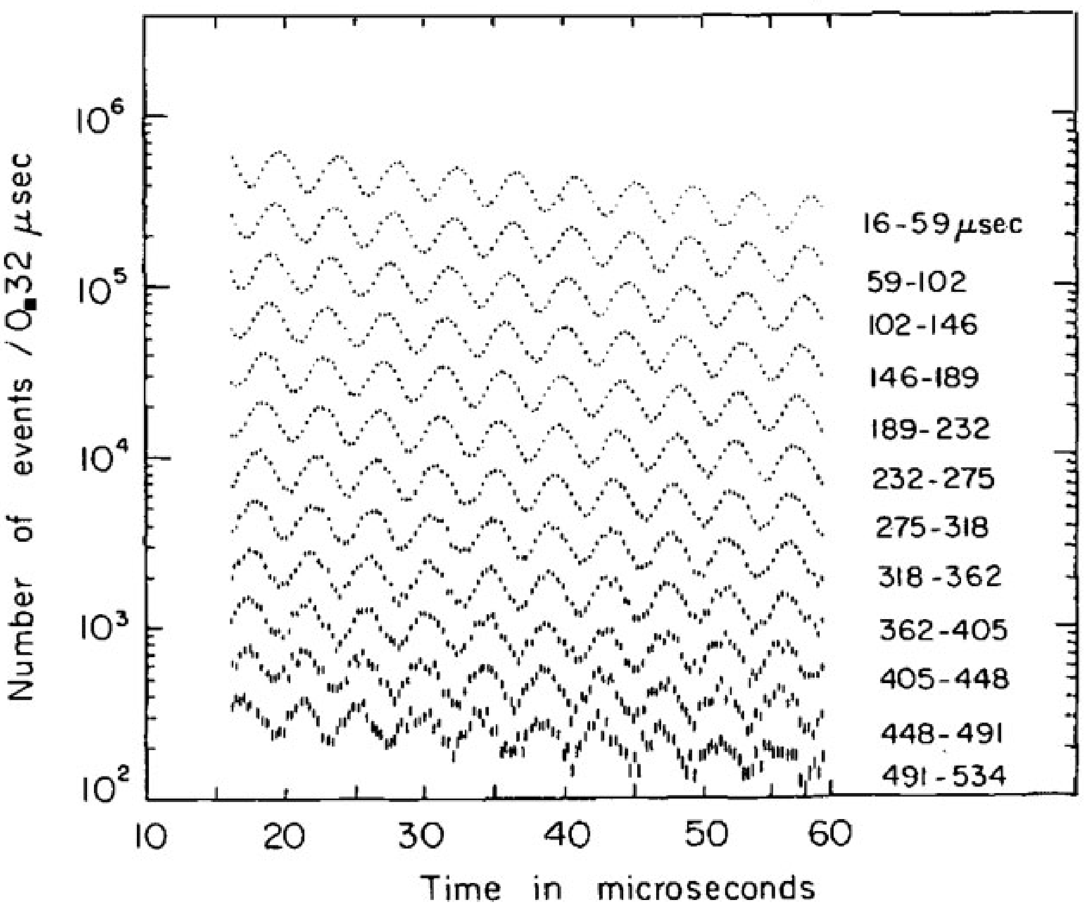
Conceptual layout of the (g-2) experiment



View of BNL (g-2) experiment



Typical count rates for electrons from muon decay



Count rate of electrons from muon decay: periodicity gives the precession frequency of the muon and hence g-2