Monte-Carlo-Simulation Ising-Modell

Kritische Exponenten des Ising-Modells in $d = 3$

$$\alpha = 0.110(1); \quad \beta = 0.3265(3); \quad \gamma = 1.2372(5);$$
$$\delta = 4.789(2); \quad \nu = 0.6301(4); \quad \eta = 0.0364(5); \quad \omega = 0.84(4)$$

Pelissetto, Vicari, Physics Reports 386, 549 (2002)

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\text{Magnetisierung} & m = m_0 \epsilon^{\beta} \\
\text{Suszeptibilität} & \chi = \chi_0 \epsilon^{-\gamma} \\
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Zum Vergleich:
mean field $d \geq 4$: $\alpha = 0, \beta = 1/2, \gamma = 1, \nu = 1/2$
$d = 2$: $\alpha = 0, \nu = 1, \beta = \frac{1}{8}, \gamma = \frac{7}{4}, \delta = 15$

Finite-size scaling (FSS)

Aim

Extrapolation of critical behavior from finite system

(non-singular)

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Hence, numerical results for a hypercubic lattice with volume $V = L^d$ depend only on $\frac{L}{\xi}$ close to T_c.

Scaling, renormalization group (RG) s.

Scaling

To leading order in $1/L$, singular part (s) of F is (Wilson's RG):

$$f^{(s)}(L,T) = \frac{1}{V}F^{(s)}(L,T) \approx \frac{1}{V}\tilde{\Psi}\left(\frac{L}{\xi(T)}\right), \quad \text{where}$$
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Rescaling to microscopic length scale L_0:

$$\Big(\frac{L}{\xi}\frac{\xi_0}{L_0}\Big)^{\frac{1}{\nu}} = \epsilon \Big(\frac{L}{L_0}\Big)^{\frac{1}{\nu}} \qquad (3)$$

Scaling

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All exponents and dependence of $\tilde{\Psi}$ on L/ξ are universal (only depend on universality class which in turn dependes on dimensionality and type of order parameter).
It is possible to derive scaling equations for the observables: (setting $L_0 \equiv 1$):

$$\begin{aligned} m(L,T) &= \tilde{m}(\epsilon L^{\frac{1}{\nu}})L^{-\frac{\beta}{\nu}} && (4) \\ \chi(L,T) &= \tilde{\chi}(\epsilon L^{\frac{1}{\nu}})L^{\frac{\gamma}{\nu}} && (5) \\ C(L,T) &= \tilde{C}(\epsilon L^{\frac{1}{\nu}})L^{\frac{\alpha}{\nu}} && (6) \end{aligned}$$

To arrive at these equations, relations between the exponents were used:

$\alpha + 2\beta + \gamma = 2$	Rushbrooke	$\gamma = \beta(\delta - 1)$	Widom
$d\nu = 2 - \alpha$	Josephson	$\gamma = \nu(2 - \eta)$	Fisher

Finite size scaling

Note

1. *scaling laws only true asymptotically for* $\epsilon \to 0$ *and* $B \to 0$
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Practical applications of scaling:
To test critical exponents plot

$$\tilde{m}_L(x), \text{ where } \tilde{m}_L = m_L L^{\frac{\beta}{\nu}};\ x = \epsilon L^{\frac{1}{\nu}} \tag{7}$$

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Curves asymptotically collapse

$$\tilde{m}(x) = \lim_{L\to\infty} \tilde{m}_L(x) \tag{8}$$

seperately for $T < T_c$ and $T > T_c$.

Finite size scaling

Problem

1. *Replacing* $T \to \epsilon = |1 - \frac{T}{T_c}|$ *requires knowledge of* T_c *(non-universal)*
2. *exponents unknown or not known accurately*

Finite size scaling

Solution:
Determine T_c from observables which are not renormalized by L. For example:

$$\langle M^2 \rangle \propto (\underset{\substack{\uparrow \\ M=L^d m}}{L^d} L^{-\frac{\beta}{\nu}})^2 \text{ und } \langle M^4 \rangle \propto (L^d L^{-\frac{\beta}{\nu}})^4 \tag{9}$$

ratio not renormalized

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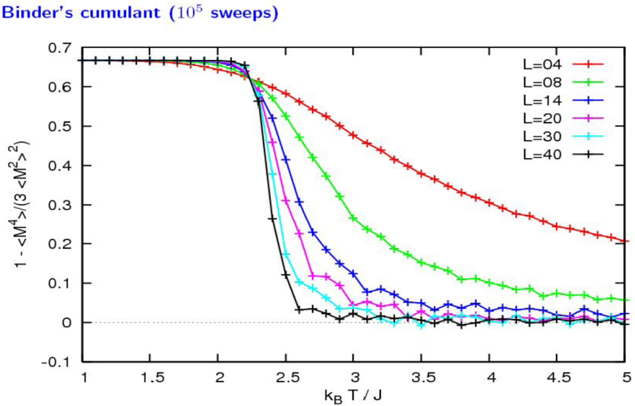
or *Binder cumulant* $U_4 = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$ with $U_4(L,T) = \tilde{U}_4(x)$
Plotting curves $U_4(T)$ for different L, these curves cross asymptotically at $T = T_c$.

Finite size scaling

For example: 2D Ising-Modell



Asymptotic behavior of U_4 $\overset{L\to\infty}{\longrightarrow} \begin{cases} \frac{2}{3} & \text{für} \quad T < T_c \\ U^* \approx 0.61 & \text{"} \quad T = T_c \\ 0 & \text{"} \quad T > T_c \end{cases}$

Critical exponents:
As soon as T_c is known, exponents can be determined by logarithmic plot, here for β