

Monte-Carlo-Simulation Ising-Modell

Kritische Exponenten des Ising-Modells in $d = 3$

$$\alpha = 0.110(1); \quad \beta = 0.3265(3); \quad \gamma = 1.2372(5);$$

$$\delta = 4.789(2); \quad \nu = 0.6301(4); \quad \eta = 0.0364(5); \quad \omega = 0.84(4)$$

Pelissetto, Vicari, Physics Reports 386, 549 (2002)

$$\left. \begin{array}{ll} \text{Magnetisierung} & m = m_0 \epsilon^\beta \\ \text{Suszeptibilität} & \chi = \chi_0 \epsilon^{-\gamma} \\ \text{Spezifische Wärme} & C = C_0 \epsilon^{-\alpha} \\ \text{Korrelationslänge} & \xi = \xi_0 \epsilon^{-\nu} \\ \text{Magnetisierung bei } T_c & m \propto B^{\frac{1}{\delta}} \end{array} \right\} \epsilon = \left| 1 - \frac{T}{T_c} \right|$$

Zum Vergleich:

mean field $d \geq 4$: $\alpha = 0, \beta = 1/2, \gamma = 1, \nu = 1/2$

$d = 2$: $\alpha = 0, \nu = 1, \beta = \frac{1}{8}, \gamma = \frac{7}{4}, \delta = 15$

Finite-size scaling (FSS)

Aim

Extrapolation of *critical behavior* from *finite system*
(non-singular)

Hypothesis:

Relevant for singular behavior in F close to 2. order phase transition is *correlation length* ξ ,
not microscopic length scales (e.g., lattice distance).

Hence, numerical results for a hypercubic lattice with volume $V = L^d$ depend only on $\frac{L}{\xi}$ close to T_c .

Scaling, renormalization group (RG) s.



Scaling

To leading order in $1/L$, **singular part (s) of F** is (Wilson's RG):

$$f^{(s)}(L, T) = \frac{1}{V} F^{(s)}(L, T) \approx \frac{1}{V} \tilde{\Psi}\left(\frac{L}{\xi(T)}\right), \quad \text{where} \quad (1)$$

$$\xi(T) = \xi_0 \epsilon^{-\nu}; \quad \epsilon = \left|1 - \frac{T}{T_c}\right|$$

\Rightarrow **dimensionless length:**

$$\frac{L}{\xi} = \xi_0^{-1} \epsilon^{\nu} L \quad (2)$$

Rescaling to microscopic length scale L_0 :

$$\left(\frac{L}{\xi} \frac{\xi_0}{L_0}\right)^{\frac{1}{\nu}} = \epsilon \left(\frac{L}{L_0}\right)^{\frac{1}{\nu}} \quad (3)$$

Scaling

All exponents and dependence of $\tilde{\Psi}$ on L/ξ are universal (only depend on universality class which in turn depends on dimensionality and type of order parameter).

It is possible to derive scaling equations for the observables: (setting $L_0 \equiv 1$):

$$m(L, T) = \tilde{m}(\epsilon L^{\frac{1}{\nu}}) L^{-\frac{\beta}{\nu}} \quad (4)$$

$$\chi(L, T) = \tilde{\chi}(\epsilon L^{\frac{1}{\nu}}) L^{\frac{\gamma}{\nu}} \quad (5)$$

$$C(L, T) = \tilde{C}(\epsilon L^{\frac{1}{\nu}}) L^{\frac{\alpha}{\nu}} \quad (6)$$

To arrive at these equations, relations between the exponents were used:

$$\alpha + 2\beta + \gamma = 2 \text{ Rushbrooke} \quad \gamma = \beta(\delta - 1) \text{ Widom}$$

$$d\nu = 2 - \alpha \quad \text{Josephson} \quad \gamma = \nu(2 - \eta) \text{ Fisher}$$

Finite size scaling

Note

- ① *scaling laws only true asymptotically for $\epsilon \rightarrow 0$ and $B \rightarrow 0$
in practice corrections can be significant*
- ② *in general there is an additional (non-singular) regular part*

Practical applications of scaling:

To test critical exponents plot

$$\tilde{m}_L(x), \text{ where } \tilde{m}_L = m_L L^{\frac{\beta}{\nu}}; \quad x = \epsilon L^{\frac{1}{\nu}} \quad (7)$$

Curves asymptotically collapse

$$\tilde{m}(x) = \lim_{L \rightarrow \infty} \tilde{m}_L(x) \quad (8)$$

seperately for $T < T_c$ and $T > T_c$.

Finite size scaling

Problem

- 1 Replacing $T \rightarrow \epsilon = |1 - \frac{T}{T_c}|$ requires knowledge of T_c (non-universal)
- 2 exponents unknown or not known accurately

Finite size scaling

Solution:

Determine T_c from observables which are **not** renormalized by L . For example:

$$\langle M^2 \rangle \propto (\underbrace{L^d}_{M=L^d m} L^{-\frac{\beta}{\nu}})^2 \text{ und } \langle M^4 \rangle \propto (L^d L^{-\frac{\beta}{\nu}})^4 \quad (9)$$

ratio not renormalized

$$\frac{\langle M^4 \rangle}{\langle M^2 \rangle^2} \propto (L^{d-\frac{\beta}{\nu}})^{4-2 \cdot 2} = L^0 \quad (10)$$

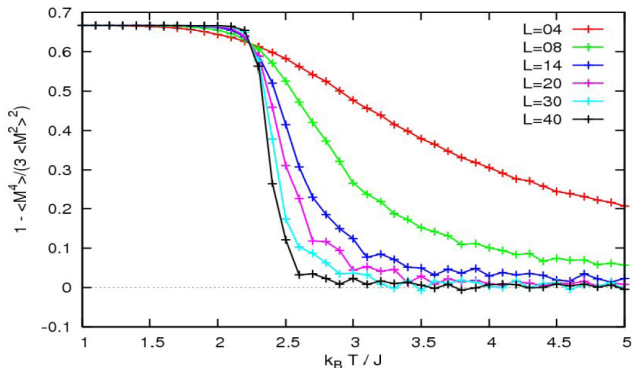
or **Binder cumulant** $U_4 = 1 - \frac{\langle M^4 \rangle}{3\langle M^2 \rangle^2}$ with $U_4(L, T) = \tilde{U}_4(x)$

Plotting curves $U_4(T)$ for different L , these curves cross asymptotically at $T = T_c$.

Finite size scaling

For example: 2D Ising-Modell

Binder's cumulant (10^5 sweeps)



Asymptotic behavior of $U_4 \xrightarrow{L \rightarrow \infty} \begin{cases} U^* \approx \frac{2}{3} \approx 0.61 & \text{für } T < T_c \\ 0 & \text{für } T = T_c \\ 0 & \text{für } T > T_c \end{cases}$

Critical exponents:

As soon as T_c is known, exponents can be determined by logarithmic plot, here for β

